Complex manifolds NGA course 2023

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Lecture Notes

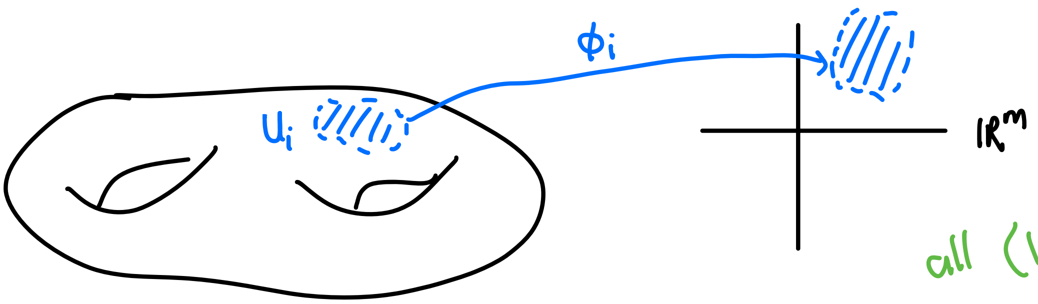
Main reference: Le Floch, A Brief Introduction to Berezin-Toeplitz Operators on Compact Kähler Manifolds, chaps 1-4

For smooth manifolds (chapter 1): • Looijenga, Notes on smooth manifolds

1.1. Smooth manifolds and smooth maps

Definition An <u>m-dimensional smooth atlas</u> $(U_i, \phi_i)_{i\in I}$ on a topological space M is an open cover (U_i) of M together with <u>local charts</u> (homeomorphisms)

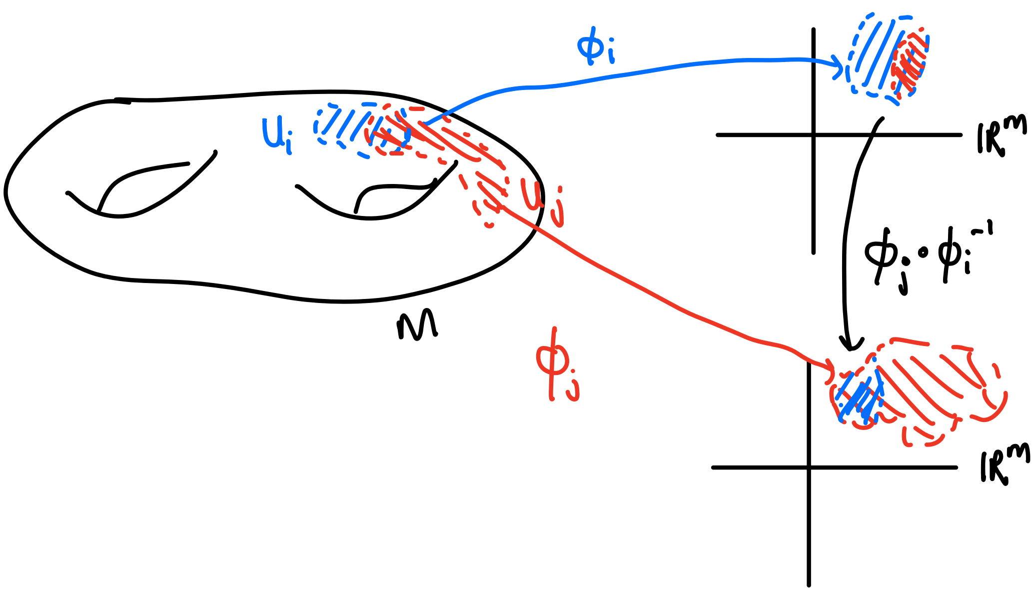
$$\phi_i : U_i \xrightarrow{\cong} \text{open subset of } \mathbb{R}^m$$



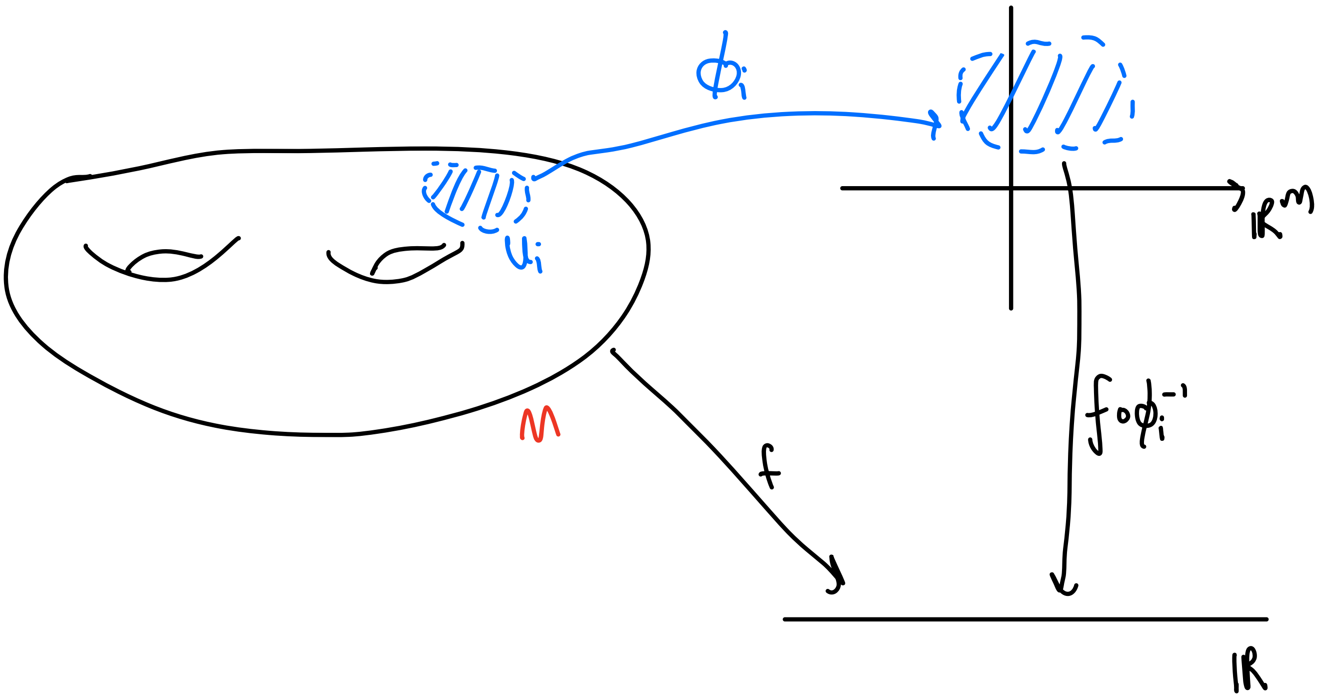
all (higher) partial derivatives exist

satisfying, for all $U_i \cap U_j \neq \emptyset$,

$$\phi_j \circ \phi_i^{-1} \text{ is a } \underline{\text{smooth}} \text{ map (between open subsets of } \mathbb{R}^m)$$



An atlas on M allows us to say when a function $f: M \longrightarrow \mathbb{R}$ is <u>smooth</u>: namely, we demand, for all charts (U_i, ϕ_i), that $f \circ \phi_i^{-1}$ is a smooth map (from an open subset of $\mathbb{R}^m$ to $\mathbb{R}$, where we know what that means).



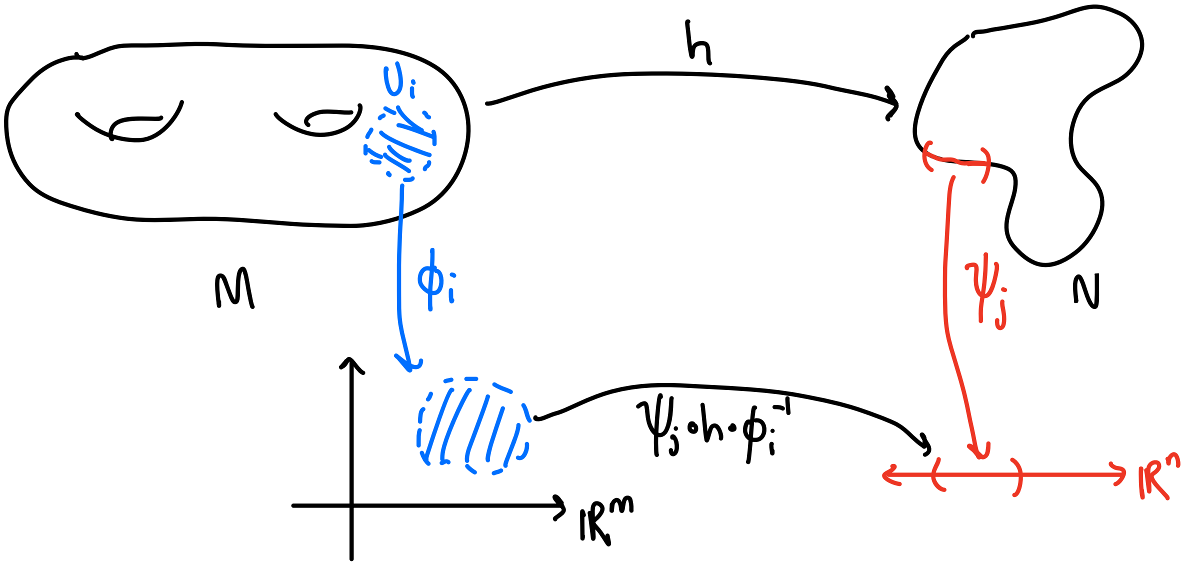
<u>Definition</u> Two smooth atlases $(U_i, \phi_i)_{i \in I}$ and $(V_k, \psi_k)_{j \in J}$ on M are <u>equivalent</u> if they agree on which functions $f: M \longrightarrow \mathbb{R}$ are smooth.

<u>Definition</u> An <u>m-dimensional smooth manifold</u> is a (Hausdorff, 2nd countable) topological space M equipped with an equivalence class of an m-dimensional smooth atlas.

Definition A map $h: M \longrightarrow N$ between smooth manifolds $(M, (U_i, \phi_i))$ and $(N, (V_j, \psi_j))$ is smooth if, for all i, j,

$$\psi_j \circ h \circ \phi_i^{-1} : \text{open subset of } \mathbb{R}^m \longrightarrow \text{open subset of } \mathbb{R}^n$$

is smooth.



A smooth map $h: M \longrightarrow N$ is called a diffeomorphism if h^{-1} exists and is smooth.

Example 1 $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ is a smooth manifold.

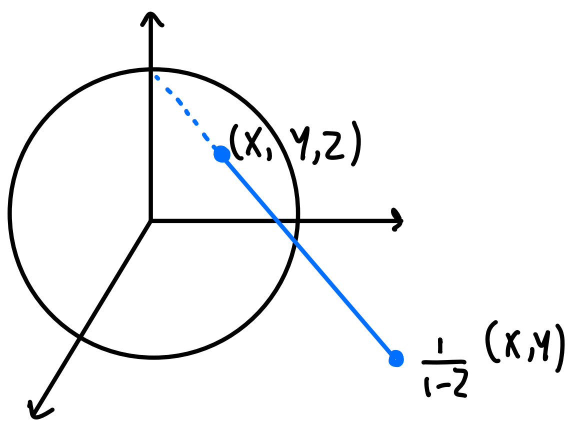
Smooth atlas : • $U_N = S^2 \setminus \{(0,0,-1)\}$

$$\phi_N : U_N \xrightarrow{\cong} \mathbb{R}^2 \qquad \text{stereographic projection from north pole}$$

$$(x,y,z) \longmapsto \frac{1}{1-z}(x,y)$$

$$\frac{1}{1+x^2+y^2}\left(2x, 2y, x^2+y^2-1\right) \longleftarrow (x,y)$$

Exercise 1.a.



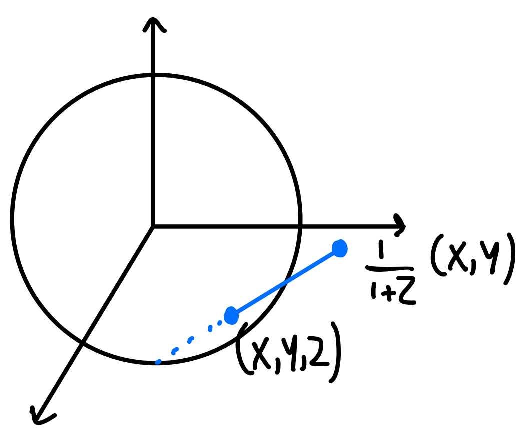
• $U_S = S^2 \setminus \{0,0,1\}$

$$\phi_S : U_S \xrightarrow{\cong} \mathbb{R}^2$$

$$(x,y,z) \longmapsto \frac{1}{1+z}(x,y)$$

$$? \longleftarrow (x,y)$$

Exercise 1.b.



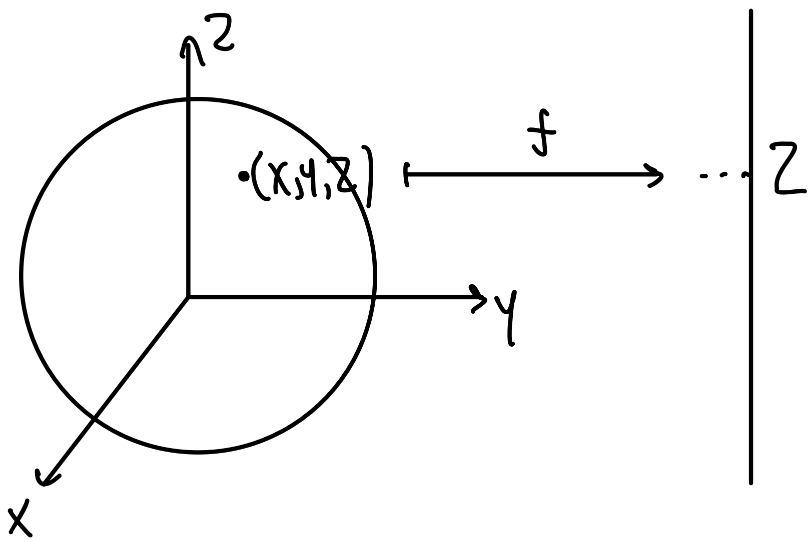
To check it is a smooth atlas, must check if the coordinate change map

$$\phi_S \circ \phi_N^{-1} : \mathbb{R}^2 \setminus \{(0,0)\} \xrightarrow{\cong} \mathbb{R}^2 \setminus \{(0,0)\}$$

is smooth. (Exercise 1c)

As an example of a smooth function on S^2, consider

$$f : S^2 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longmapsto z$$

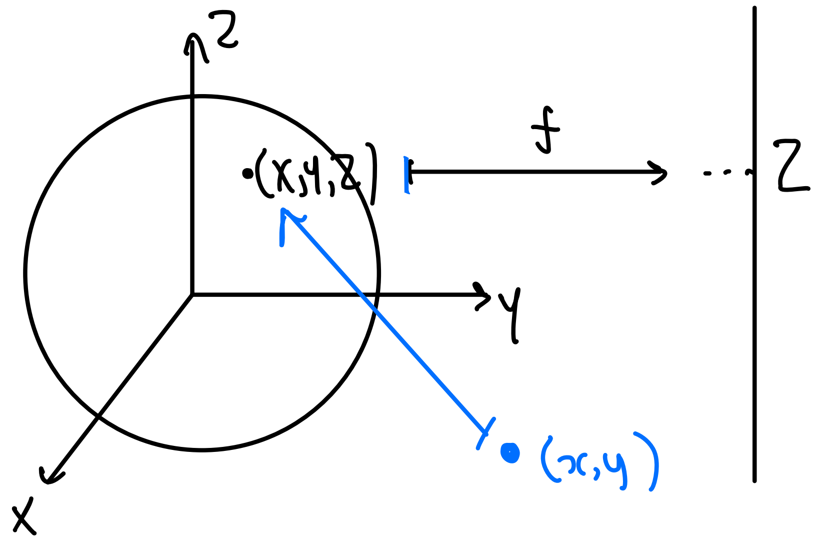


Is it smooth? On the chart (U_N, ϕ_N), we compute:

$$f_N(x,y) := f \circ \phi_N^{-1}(x,y)$$

$$(x,y) \xmapsto{\phi_N^{-1}} \frac{1}{1+x^2+y^2}\left(2x, 2y, x^2+y^2-1\right) \xmapsto{f} \frac{x^2+y^2-1}{1+x^2+y^2}$$

Is this a smooth map? Yes.



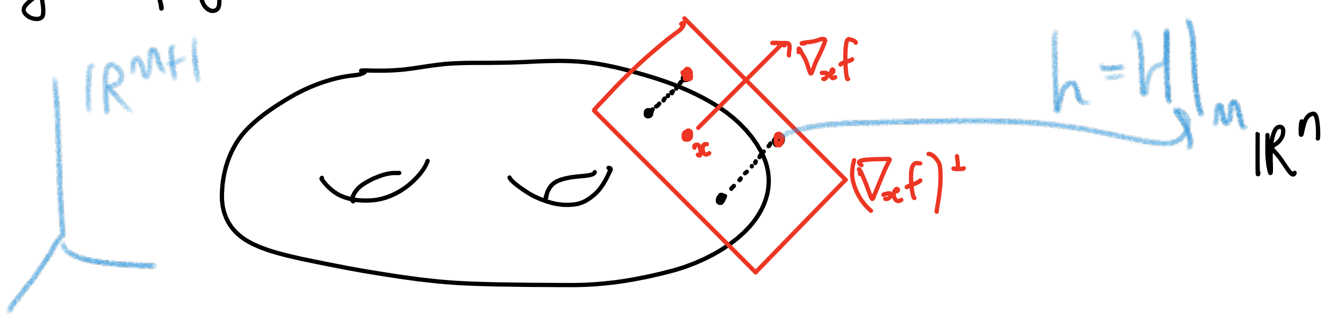
$$f_N : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

Exercise 2 Check that the formulas for ϕ_N, ϕ_N^{-1}, ϕ_S are correct and compute ϕ_S^{-1}. Check that the transition function (coordinate change map) $\phi_N \circ \phi_S^{-1}$ is smooth [and also its inverse].

Example 2 (Will prove later) Given a smooth function

$$f : \mathbb{R}^{m+1} \longrightarrow \mathbb{R},$$

the m-dimensional hypersurface $M := f^{-1}(c)$ will naturally be a smooth manifold precisely when $\nabla_x f \neq 0$ for all $x \in M$. A chart near $x \in M$ is defined by orthogonal projection onto $(\nabla_x f)^{\perp}$ (the <u>tangent space</u> at x ... see later):



Moreover, the atlas above has the following property: a map

$$h: M \longrightarrow \mathbb{R}^n$$

will be smooth precisely when h is the restriction of a smooth map

$$H: U \longrightarrow \mathbb{R}^n$$

where $U \subseteq \mathbb{R}^{m+1}$ is an open neighborhood of M in $\mathbb{R}^{m+1}$.

Exercise 3. Prove this.

a) eg. can express S^2 as a hypersurface:

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longmapsto x^2+y^2+z^2$$
$$S^2 := f^{-1}(1)$$

We have $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$= (2x, 2y, 2z)$$

$\neq 0$ for all points $(x,y,z) \in S^2$

$\therefore$ S^2 naturally inherits a smooth atlas.

Also, the properties of this atlas imply that eg.

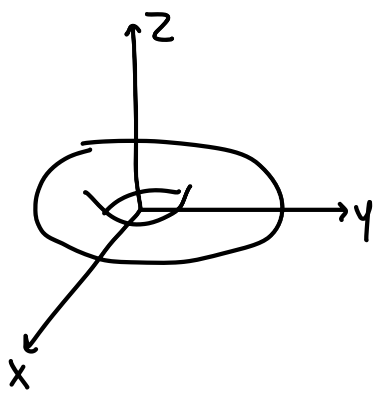
$$h: S^2 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longmapsto x^2 - \cos(y)$$

is a smooth map.

Exercise 4. Why is h a smooth map?

b) The 2-torus $T^2 \subseteq \mathbb{R}^3$ can be expressed via the equation

$$\left(2-\sqrt{x^2+y^2}\right)^2+z^2=1$$



Exercise 5. a) Check that T^2 is naturally a smooth manifold.

b) Construct an explicit, non-constant smooth map $h: T^2 \longrightarrow S^2$ and check it is smooth.

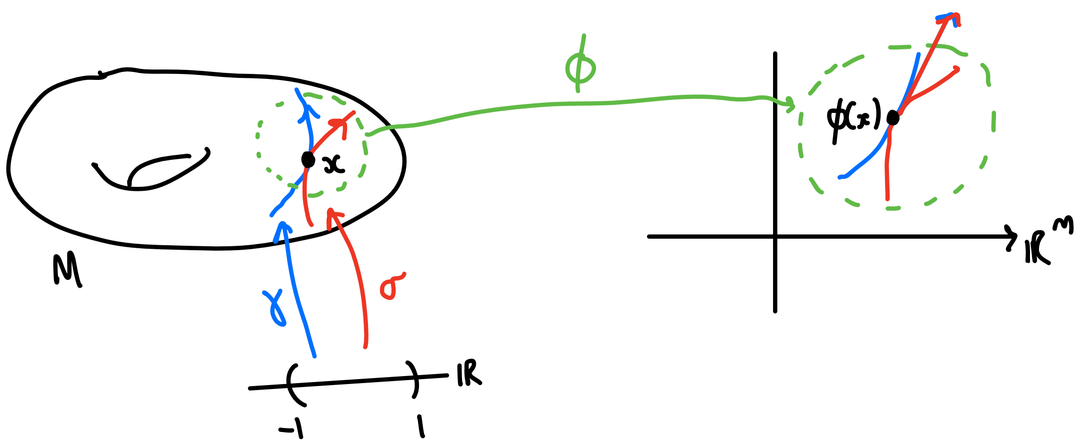
1.2. Tangent spaces

Definition Let M be a smooth manifold and $x \in M$. The <u>tangent space T_xM</u> is the set of equivalence classes of smooth curves

$$\gamma : (-1,1) \longrightarrow M, \quad \gamma(0) = x.$$

Two curves γ and σ are <u>equivalent</u> if, in some chart (U, ϕ) around x,

$$\left.\frac{d}{dt}\right|_{t=0} \underbrace{\phi \circ \gamma}_{\text{a curve in } \mathbb{R}^m} = \left.\frac{d}{dt}\right|_{t=0} \underbrace{\phi \circ \sigma}_{\text{a curve in } \mathbb{R}^m}$$



Exercise 1. Check that this equivalence relation is actually an equivalence relation. (reflexive, symmetric, transitive)

Note that a chart (U, ϕ) gives us a bijection

$$\begin{aligned} T_xM &\xrightarrow{\cong} \mathbb{R}^m \\ [\gamma] &\longmapsto \left.\frac{d}{dt}\right|_{t=0} \phi \circ \gamma \end{aligned}$$

We equip T_xM with the structure of a <u>real vector space</u> by transporting it over from $\mathbb{R}^m$ via this bijection, i.e. we set:

$$k.[\gamma] := [\phi^{-1}(k.\phi\circ\gamma)]$$

$$[\gamma]+[\sigma] := [\phi^{-1}(\phi\circ\gamma + \phi\circ\sigma)]$$

<u>Exercise 2.</u> Check that this vector space structure on T_xM does not depend on the chart (U,ϕ).

<u>Example</u> If $M \subseteq \mathbb{R}^{m+1}$ is the hypersurface of a smooth map

$$f: M \longrightarrow \mathbb{R}$$

$$(\text{i.e. } M = f^{-1}(c) \text{ for some } c)$$

then we have a canonical linear identification:

$$T_xM \cong \{ v \in \mathbb{R}^{m+1} : \nabla_x f \cdot v = 0 \}$$

<u>Exercise 3.</u> Prove this.

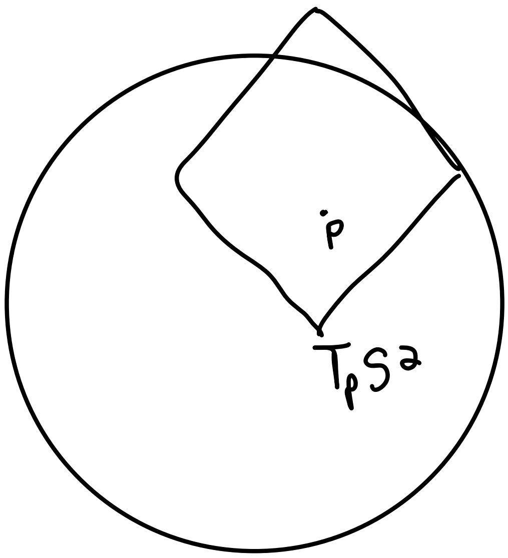
eg. For S^2, we have

$$T_p S^2 \cong \{ v \in \mathbb{R}^3 : p \cdot v = 0 \}$$

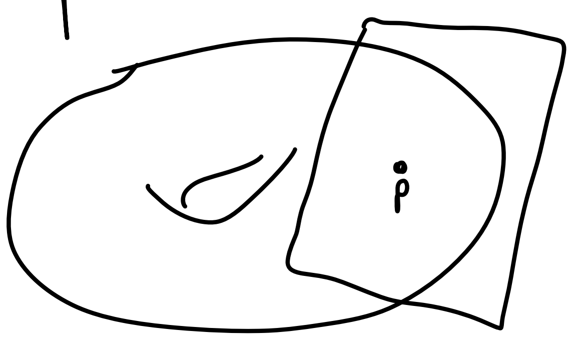
because $p = (x, y, z)$

$\nabla_p f = (2x, 2y, 2z)$

so $p \cdot v = 0 \iff \nabla_p f \cdot v = 0$



For T^2:



$T_p T^2$

Lemma i) A smooth map $f: M \longrightarrow N$ between smooth manifolds determines, for every $x \in M$, a linear map

or just f_* for short

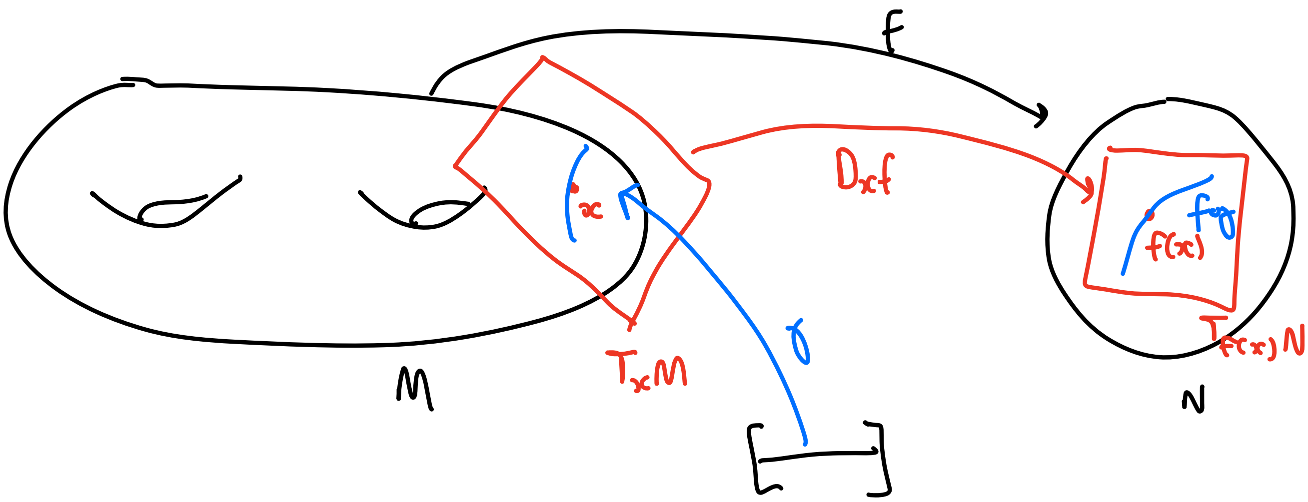
$$D_x f : T_x M \longrightarrow T_{f(x)} N$$

defined by

$$[\gamma] \longmapsto [f \circ \gamma]$$

ii) (Chain rule) If $g: N \longrightarrow P$ is another smooth map, then

$$D_x(g \circ f) = D_{f(x)}(g) \circ D_x(f)$$



Proof i) Exercise.

ii) $$D_{f(x)}(g)\Big(D_x(f)\,([\gamma])\Big) = D_{f(x)}(g)\big([f \circ \gamma]\big)$$

$$= \big[g \circ (f \circ \gamma)\big]$$

$$= \big[(g \circ f) \circ \gamma\big]$$

$$= D_x(g \circ f)([\gamma]).$$

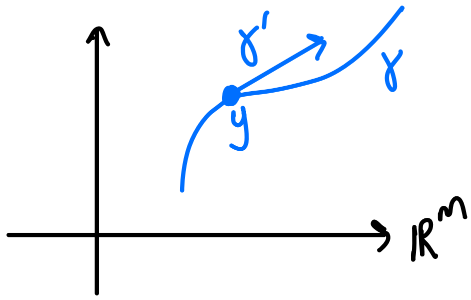
what a simple proof!

$\square$

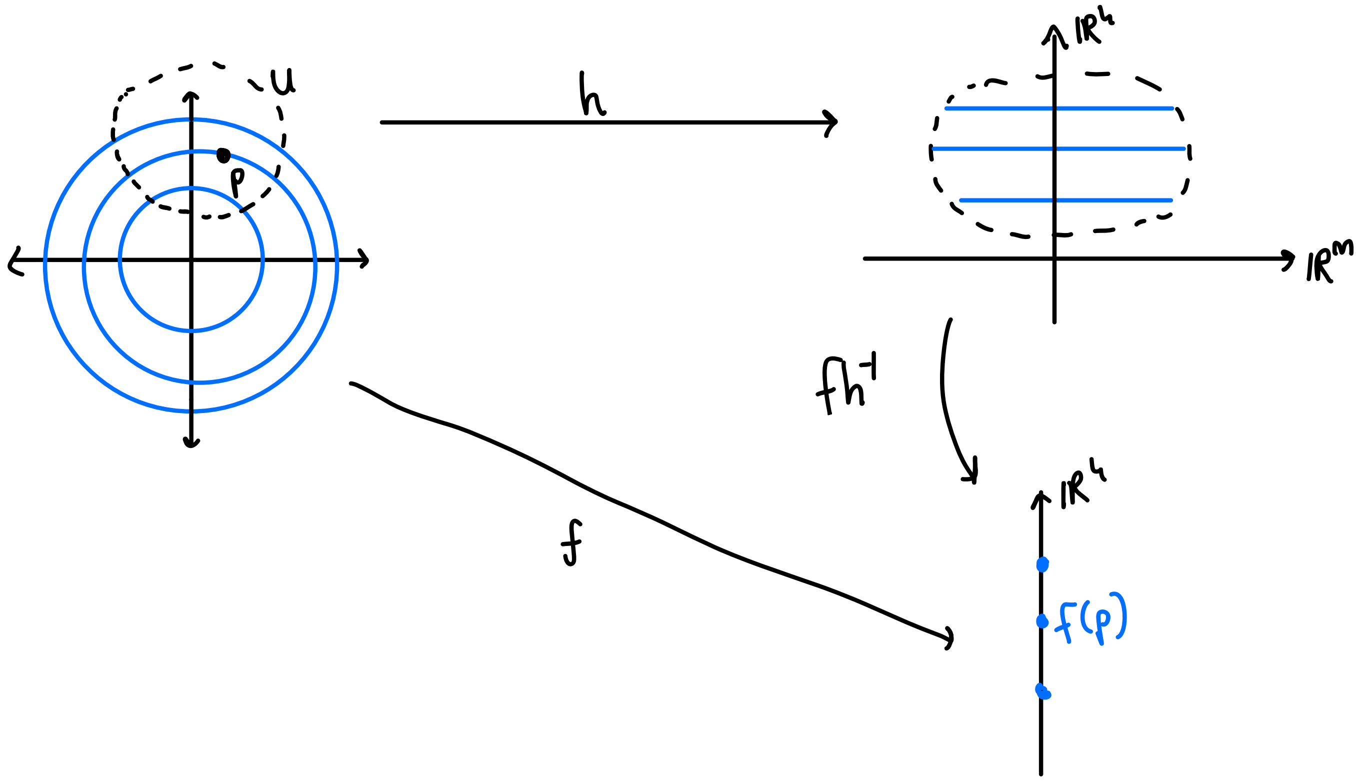
Note that when we think of $\mathbb{R}^n$ as a smooth manifold (via the "identity" atlas), then for all $y \in \mathbb{R}^n$, we can canonically identify

$$T_y\mathbb{R}^n \cong \mathbb{R}^n$$
$$[\gamma] \mapsto \frac{d}{dt}\Big|_{t=0} \gamma$$

this only makes sense because the curve γ is living in $\mathbb{R}^n$.



*(Extra) <u>The Implicit Function Theorem</u> Let $f:\mathbb{R}^{m+k}\longrightarrow\mathbb{R}^m$ be a smooth map, and suppose that at $p\in\mathbb{R}^{m+k}$, D_pf is surjective. Then there exists a diffeomorphism h of an open neighborhood of p onto an open subset of $\mathbb{R}^n$ such that fh^{-1} is on its domain the projection $\mathbb{R}^m\times\mathbb{R}^k\longrightarrow\mathbb{R}^k$.



<u>Exercise 4</u> Show that (as promised earlier) this means that for a hypersurface $M:=f^{-1}(c)$ of a smooth function $f:\mathbb{R}^{m+1}\longrightarrow\mathbb{R}$ (where $\nabla_pf\neq 0$ for all $p\in M$), this equips each tangent space T_pM with a local chart for M.

1.3. The tangent bundle

The collection of all the tangent spaces of an m-dimensional manifold forms a $2m$-dimensional manifold!

<u>Definition</u> The <u>tangent bundle</u> TM of an m-dimensional smooth manifold M is the set

$$TM := \{ (x,v) \ : \ x \in M, \ v \in T_x M \}$$

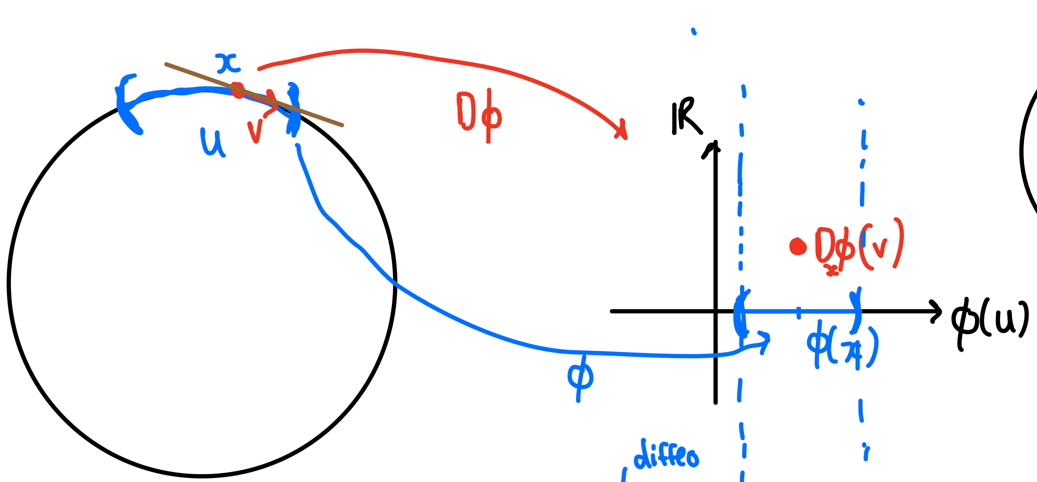
equipped with the following smooth atlas:

Let (U, ϕ) be a chart for M. We get a chart

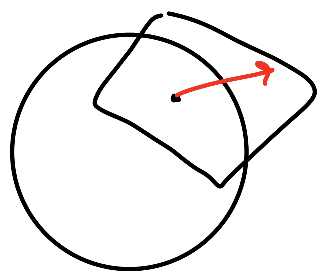
$$D\phi \ : \ TU \longrightarrow \phi(U) \times \mathbb{R}^m$$
$$(x,v) \longmapsto (\phi(x), \underbrace{D_x\phi(v)}_{\text{lives in } T_{\phi(x)}\mathbb{R}^m \cong \mathbb{R}^m.})$$

$\phi : U \longrightarrow \mathbb{R}^m$

$D_x\phi : T_xU \longrightarrow T_{\phi(x)}\mathbb{R}^m \cong \mathbb{R}^m$



S^1



$TS^1 \cong S^1 \times \mathbb{R}$

$TS^2 \ncong S^2 \times \mathbb{R}^2$

Exercise 1. I didn't say what the topology on TM is. We define a set $\Omega \subseteq TM$ to be open if for any chart (U, ϕ) of M, $D\phi(\Omega)$ is open in $\mathbb{R}^{2m}$. Check that this indeed defines a topology, and that the chart maps $D\phi : TU \longrightarrow \phi(U) \times \mathbb{R}^m$ are homeomorphisms.

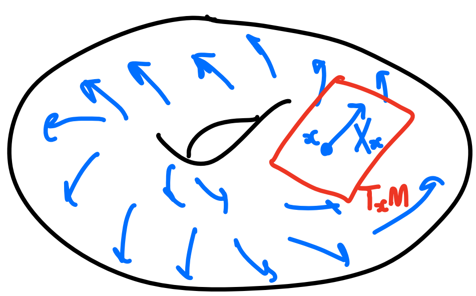
With respect to this smooth atlas, the projection map

$$\pi : TM \longrightarrow M \qquad x \in M$$
$$(x, v) \longmapsto x \qquad v \in T_x M$$

is smooth. (Exercise 2. : Check!)

Definition A smooth vector field on M is a smooth map $X : M \to TM$ which is a section of π, i.e. $\pi \circ X = \mathrm{id}_M$, i.e.

$$X_x \in T_x M \quad \text{for all } x \in M.$$



__Example__ Let (U,ϕ) be a chart of M, written as:

$$\begin{aligned} \phi: U &\xrightarrow{\cong} \phi(U) \subseteq \mathbb{R}^m \\ p &\longmapsto (x_1, \cdots, x_m) \end{aligned}$$

Then for each $p \in U$, we get the __coordinate tangent vectors__

$$\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_m}\right|_p \in T_pM$$

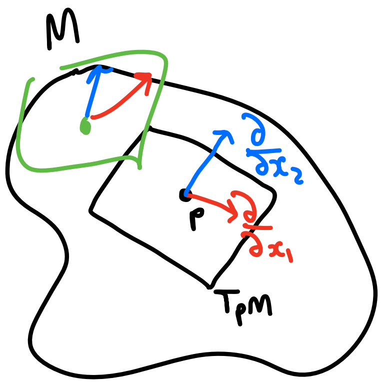
Standard basis $(0, \cdots, 1, \cdots, 0)$ of $\mathbb{R}^m$

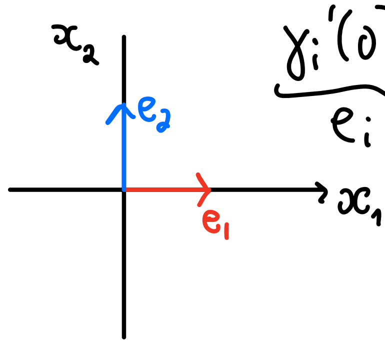
defined as

$$\left.\frac{\partial}{\partial x_i}\right|_p := D_0\phi^{-1}(e_i) = [\phi^{-1}(\gamma_i)]$$

$$\gamma_i(t) = (0, \cdots, t, \cdots, 0)$$

$$\underbrace{\gamma_i'(0)}_{e_i} = (0, \cdots, 1, \cdots 0)$$

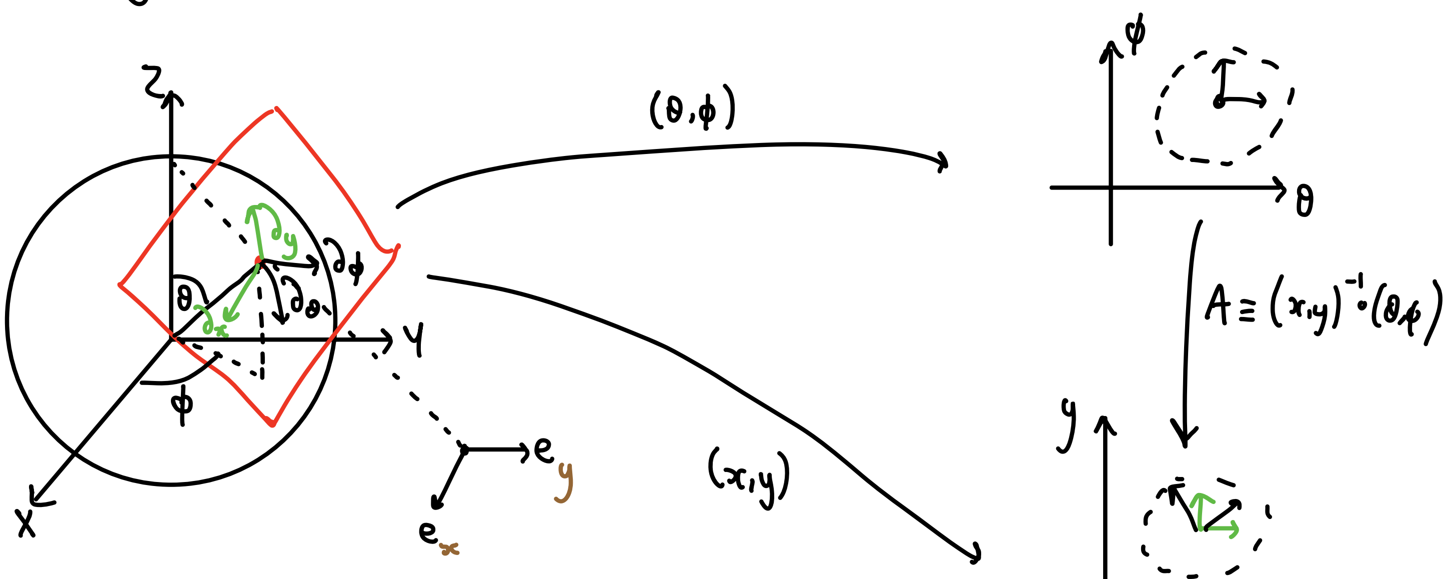




These form a __basis__ for T_pM, and each $\frac{\partial}{\partial x_i}$ is a smooth vector field on $U \subseteq M$.

> __Lemma__ A vector field X on M is smooth if and only if, when we expand it relative to the $\frac{\partial}{\partial x_i}$ basis coming from a coordinate chart U, i.e. write
>
> $$X_p = X_1(p) \left.\frac{\partial}{\partial x_1}\right|_p + \cdots + X_m(p) \left.\frac{\partial}{\partial x_m}\right|_p$$
>
> then each component function X_i is smooth on U.

<u>Example</u> Express the "latitude/longitude" $\partial_\theta, \partial_\phi$ vector fields relative to the "stereographic projection from north pole" ∂_x, ∂_y vector fields.



<u>Method 1</u>

In the (x,y) coordinate system, we must compute $A_*(\partial_\theta)$, $A_*(\partial_\phi)$.

We know:

$$X = \sin\theta \cos\phi$$
$$Y = \sin\theta \sin\phi \quad \Rightarrow$$
$$Z = \cos\theta$$

$$\Rightarrow \quad (x,y) = \frac{1}{1-Z}(X,Y)$$
$$\frac{1}{1-\cos\theta}\left(\sin\theta\cos\phi, \ \sin\theta\sin\phi\right)$$

Therefore the curve $e_\theta(t)$ in the (θ,ϕ) system given by

$$\theta(t) = \theta_0 + t \quad , \ \phi(t) = \phi_0$$

becomes the following curve in the (x,y)-system:

$$x(t) = \frac{\sin(\theta_0+t)\cos\phi_0}{1-\cos(\theta_0+t)} \quad , \quad y(t) = \frac{\sin(\theta_0+t)\sin\phi_0}{1-\cos(\theta_0+t)}$$

$$\left.\frac{d}{dt}\right|_{t=0} (x(t), y(t)) = \left.\left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}\right)\right|_{(\theta_0,\phi_0)}$$

$$= \frac{(1-\cos\theta_0)\cos\theta_0\cos\phi_0 + \sin\theta_0\cos\phi_0\sin\theta_0}{(1-\cos\theta_0)^2} e_x$$

i.e. in T_pS^2,

$$\partial_\theta = \boxed{\frac{(1-\cos\theta_0)\cos\theta_0\cos\phi_0 + \sin\theta_0\cos\phi_0\sin\theta_0}{(1-\cos\theta_0)^2}} \partial_x$$

$$+ \boxed{\frac{(1-\cos\theta_0)\cos\theta_0\sin\phi_0 + \sin\theta_0\sin\phi_0\sin\theta_0}{(1-\cos\theta_0)^2}} \partial_y$$

we could/should express these in terms functions in terms of (x,y)

Method 2: (to compute ∂_ϕ) For a smooth function f,

$$\partial_\phi f = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi}$$

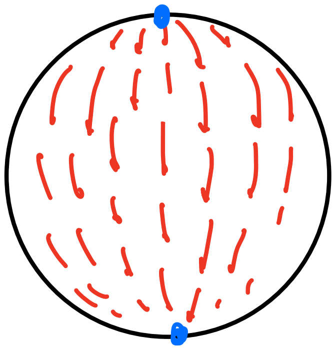
$$\partial_x \equiv \frac{\partial}{\partial x}$$

$$= \left(\frac{\partial x}{\partial \phi}\partial_x + \frac{\partial y}{\partial \phi}\partial_y\right) f$$

$$\therefore \partial_\phi = \frac{\partial x}{\partial \phi}\partial_x + \frac{\partial y}{\partial \phi}\partial_y$$

$$= \frac{-\sin\theta_0\sin\phi_0}{1-\cos\theta_0}\partial_x + \frac{\sin\theta_0\cos\phi_0}{1-\cos\theta_0}\partial_y.$$

Hairy ball theorem There is no smooth nowhere-vanishing vector field on S^2. (or on any even-dimensional sphere).



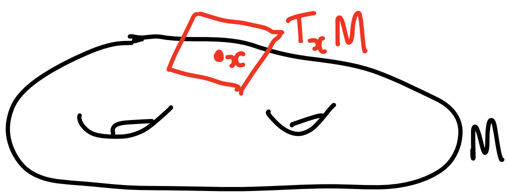
1.4 Vector bundles

The tangent bundle TM of a manifold M is not just a smooth manifold: it comes equipped with a projection map

$$\pi : TM \longrightarrow M$$

in such a way that each fiber $\pi^{-1}(x) = T_xM$ has a vector space structure.

It is a 'bundle of vector spaces' !



Definition A smooth real vector bundle of rank k over a manifold M is a collection of k-dimensional real vector spaces

$$E_x, \quad x \in M$$

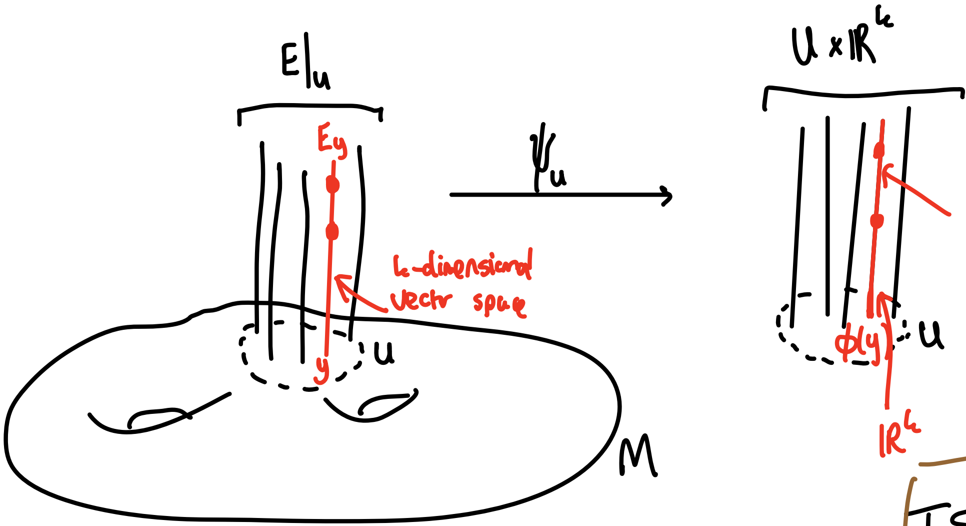
together with a smooth atlas on the total space $E := \bigsqcup_{x \in M} E_x$ such that the projection map $\pi : E \longrightarrow M$ is smooth, and such that M can be covered by open sets U where each U is equipped with a diffeomorphism

$$\psi : E|_U \xrightarrow{\cong} U \times \mathbb{R}^k$$

which commutes with the projection maps onto U, and which restricts to a linear isomorphism of vector spaces

$$\psi_x : E_x \xrightarrow{\cong} \mathbb{R}^n$$

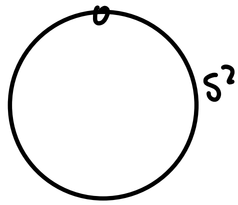
for each $x \in U$.



$TS^2 \not\cong S^2 \times \mathbb{R}^2$

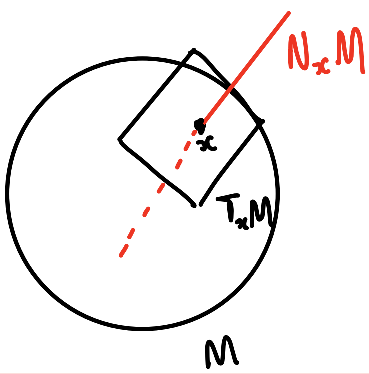
but locally it is true

$TS^2|_U \cong U \times \mathbb{R}^2$



Examples

- The tangent bundle TM of a manifold is a vector bundle of rank m.
- The trivial bundle $M \times \mathbb{R}^k$ is a rank k vector bundle.
- The normal bundle NM of a hypersurface $M \subseteq \mathbb{R}^{m+1}$ is a vector bundle of rank 1.



Exercise 1. Check this. What is a smooth atlas for NM?

If E is a smooth vector bundle over M, we write

$$C^\infty(M, E) = \{ \text{smooth sections of } E \}.$$

(Slight abuse of notation!)

1.5. Multilinear algebra

Work through Looijenga appendix A.1 - A.3 by yourself!

In summary:

- For any vector space V, we have the dual space

$$V^* := \mathrm{Hom}(V, \mathbb{R})$$

If e_i, $i=1\ldots n$ is a basis for V, then

$$e^i, \quad i=1\ldots n \qquad : e^i(e_j) = \delta_{ij}$$

is a basis for V^*, called the dual basis of $\{e_i\}$.

Exercise 1. Prove this.

- For any set X, we have the vector space

$$\mathbb{R}[X] := \left\{ \begin{array}{c} \text{all formal linear combinations } a_1 e_{x_1} + \cdots + a_n e_{x_n} \\ x_1, \ldots, x_n \in X \quad a_1, \ldots, a_n \in \mathbb{R}, \quad n \in \mathbb{N} \end{array} \right\}$$

- For any vector spaces V and W, we have the <u>tensor product space</u>

$$V \otimes W := \frac{\mathbb{R}[V \times W]}{I}$$

where I is the subspace spanned by vectors of the form

- $e_{(v_1+v_2,w)} - e_{(v_1,w)} - e_{(v_2,w)}$
- $e_{(v,w_1+w_2)} - e_{(v,w_1)} - e_{(v,w_2)}$
- $e_{(kv,w)} - k e_{(v,w)}$
- $e_{(v,kw)} - k e_{(v,w)}$

$v \otimes 7w = 7 v \otimes w$

$(v_1+v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

We write the equivalence class $[e_{(v,w)}]$ as $v \otimes w$.

Note: Not every vector in $V \otimes W$ is of the form $v \otimes w$! (a general element will be a <u>linear combination</u> of such elements).

<u>Exercise 2.</u> Let $V = \text{span}\{e_1, e_2\}$ Consider

$$v = e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes V.$$

Prove that v cannot be written in the form $v_1 \otimes v_2$ for $v_1, v_2 \in V$.

If $e_1, \cdots, e_m$ is a basis for V and $f_1, \cdots, f_n$ is a basis for W, then

$$e_i \otimes f_j \qquad i = 1 \ldots m, \; j = 1 \ldots n$$

is a basis for $V \otimes W$.

Exercise 3. Prove this.

Contrast: $V \times W = \{(v,w) : v \in V, w \in W\}$ Basis $(e_i, 0), (0, f_j)$
$\dim(V \times W) = \dim V + \dim W$.

- For any vector space V, we have the <u>tensor algebra</u>

$$T(V) = \bigoplus_{k \geq 0} V^{\otimes k} \leftarrow \text{this means } \underbrace{V \otimes \cdots \otimes V}_{k}$$

$$= (\mathbb{R}) \oplus (V) \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

where on product vectors the multiplication is defined by

$$(v_1 \otimes v_2) \cdot (v_3 \otimes v_4 \otimes v_5) = v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5.$$

and extended to $T(V)$ by linearity.

- For any vector space V, we have the <u>exterior algebra</u>

$$\Lambda(V) = \frac{T(V)}{I} \qquad \begin{aligned} &(1+v) \cdot (w \otimes w \otimes d) \\ &= \underbrace{w \otimes w \otimes d + v \otimes w \otimes w \otimes d}_{\in I}, \end{aligned}$$

where I is the subspace spanned by vectors of the form

"a stammering tensor" $\longrightarrow$ $v_1 \otimes v_2 \otimes \cdots \otimes v_k$, $k \in \mathbb{N}$, $v_i = v_{i+1}$ for some $1 \leq i \leq k-1$.

So we have

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \Lambda^3(V) \oplus \cdots$$

$$= \mathbb{R} \oplus V \oplus \Lambda^2(V) \oplus \Lambda^3(V) \oplus \cdots$$

The equivalence class of $[v_1 \otimes \cdots \otimes v_k]$ in $\Lambda^k(V)$ is written $v_1 \wedge \cdots \wedge v_k$.

Note that $v_1 \wedge v_2 = -v_2 \wedge v_1$. $\leftarrow$ why?

If e_i, $i=1\dots m$ is a basis of V, then

$$e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k} \qquad 1\leq i_1<i_2<\cdots<i_k\leq m$$

is a basis for $\Lambda^k(V)$.

Exercise 4. Show that for any finite-dimensional vector spaces V and W, there is a canonical linear isomorphism (i.e. independent of a choice of basis)

$$V^*\otimes W \cong \mathrm{Hom}(V,W).$$

Exercise 5. If $A: V\longrightarrow W$ is a linear map, we define its k^{th} exterior power

$$\Lambda^k(A) : \Lambda^k(V)\longrightarrow \Lambda^k(W)$$

on wedge products by

$$v_1\wedge\cdots\wedge v_k \longmapsto Av_1\wedge\cdots\wedge Av_k$$

and then extend this definition to all of $\Lambda^k(V)$ by linearity. If $\dim(V)=m$, and $A: V\rightarrow V$ is a linear map, show that

$$\Lambda^m(A) = \det(A).\,\mathrm{id}_{\Lambda^m(V)}$$

This gives us a basis-free definition of the determinant!

$\dim V = m \Rightarrow \Lambda^{m+1}(V)=\{0\}$ $\qquad$ $\boxed{e_1\wedge\cdots\wedge e_m}$ basis for $\Lambda^m V$

Let $e_1, \cdots, e_m$ be a basis for V, and

$$A : V \longrightarrow V .$$

$V \circlearrowright A$ $e_1, \cdots, e_m$ m-dim

$\Lambda^k(V) \circlearrowright \Lambda^k A$ $\cdot\, e_1 \wedge \cdots \wedge e_m$

$$\boxed{\begin{aligned} \det A &= \det[A] \\ &= \sum_{\sigma \in S_m} (-1)^{\operatorname{sign}\sigma} A_{1\sigma(1)} \cdots A_{m\sigma(m)} \end{aligned}}$$

$$\begin{aligned} \Lambda^k A\,(e_1 \wedge \cdots \wedge e_m) &= \underbrace{Ae_1} \wedge \cdots \wedge Ae_m \\ &= \left(\sum_{i_1} A_{i_1 1} e_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_m} A_{i_m m} e_{i_m}\right) \\ &= \sum_{i_1, \cdots, i_m} A_{i_1 1} A_{i_2 2} \cdots A_{i_m m}\, e_{i_1} \wedge \cdots \wedge e_{i_m} \\ &\;\vdots \\ &= \sum_{\sigma \in S_m} (-1)^{\operatorname{sign}\sigma} A_{1\sigma(1)} \cdots A_{m\sigma(m)}\, e_1 \wedge \cdots \wedge e_m \end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

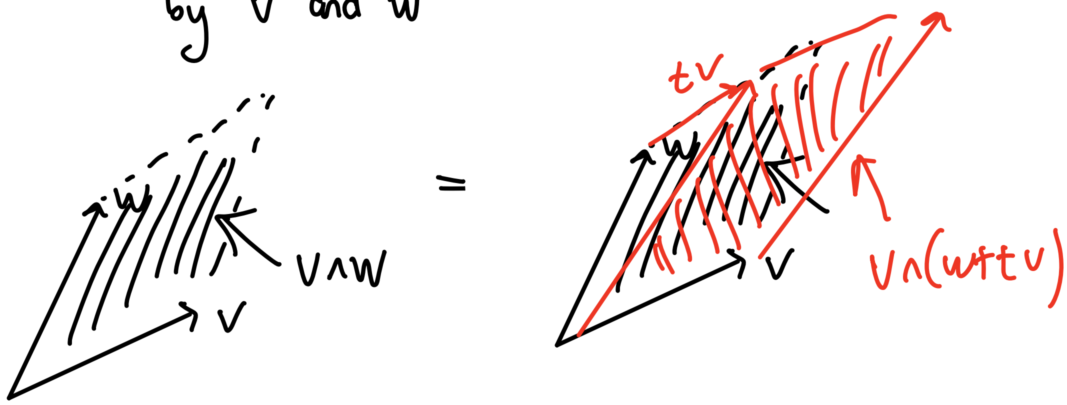
$$\begin{aligned} &Ae_1 \wedge Ae_2 \\ &= (a_{11}e_1 + a_{21}e_2) \wedge (a_{12}e_1 + a_{22}e_2) \\ &= \left[\begin{aligned} e_1 \wedge e_1 &= 0 \\ e_2 \wedge e_1 &= -e_1 \wedge e_2 \end{aligned}\right. \end{aligned}$$

Geometric interpretation of wedge products

$v \wedge w = -w \wedge v$

$$v \wedge w = v \wedge (w + tv) = v \wedge w + \underbrace{t v \wedge v}_{=0}$$

$v \wedge w$ "oriented area element spanned by v and w"



$$\Lambda^2 V = \{ \text{oriented area elements in } V \}$$

$$(\Lambda^2 V)^* = \{ \text{area functionals on } V \}$$

Moreover, for any finite-dimensional vector space V, we have a canonical linear isomorphism

$$T : \Lambda^k(V^*) \xrightarrow{\cong} (\Lambda^k V)^*$$

defined on wedge vectors by

$$T(f_1 \wedge \cdots \wedge f_k)(v_1 \wedge \cdots \wedge v_k) := \sum_{\sigma \in S_k} f_{\sigma(1)}(v_1)\, f_{\sigma(2)}(v_2) \cdots f_{\sigma(k)}(v_k).$$

For instance, suppose V is 2-dimensional. Suppose:

e_1, e_2 is a basis for V

Then

e^1, e^2 is the dual basis for V^*

Also,

$e_1 \wedge e_2$ is a basis for $\Lambda^2 V$

Let's calculate $T(e^1 \wedge e^2) \in (\Lambda^2 V)^*$. Well,

$$T(e^1 \wedge e^2)(e_1 \wedge e_2) = \underbrace{e^1(e_1)}_{=1}\, \underbrace{e^2(e_2)}_{=1} + \underbrace{e^2(e_1)}_{=0}\, \underbrace{e^1(e_2)}_{=0}$$

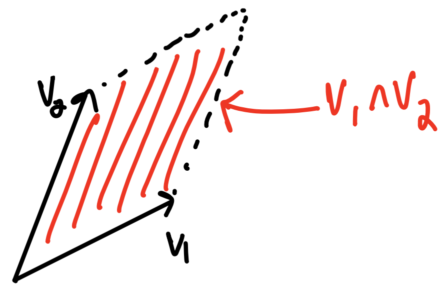
$$= 1$$

We conclude that $T(e^1 \wedge e^2)$ is the dual basis of $e_1 \wedge e_2$ in $(\Lambda^2 V)^*$!

Since

$$v_1 \wedge \cdots \wedge v_k \quad \in \Lambda^k V \qquad v_1, \cdots, v_k \in V$$

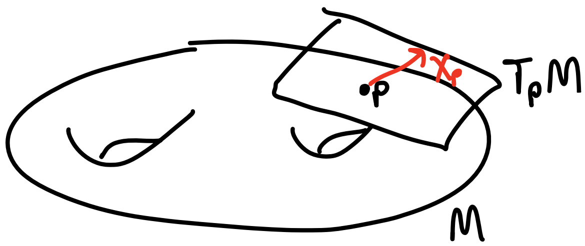
represents a k-dimensional oriented area elements in V,



we should therefore think of something in $\Lambda^k(V^*) \cong \Lambda^k(V)^*$ as a <u>measuring-stick</u> on k-dimensional oriented volume elements in V.

1.6. Differential forms

Given a smooth manifold M, recall we have the tangent bundle $TM \xrightarrow{\pi} M$:



And, a <u>smooth vector field</u> is a smooth section $X: M \longrightarrow TM$ of the tangent bundle, i.e. a smooth selection of a vector

$$X_p \in T_pM \qquad \forall\ p \in M.$$

i.e.

$$\mathrm{Vect}(M) := C^\infty(M, TM).$$

Now we have learnt functorial ways to construct new vector spaces from old: dual vector space, tensor products, wedge products. So we also have the <u>cotangent bundle</u> $T^*M \longrightarrow M$, whose fiber vector space at $p \in M$ is

$$T_p^*M := (T_pM)^* \equiv \mathrm{Hom}(T_pM, \mathbb{R}).$$

A <u>smooth 1-form</u> on M is a smooth section α of T^*M.

More generally, we have the <u>k^{th} exterior power bundle</u> $\Lambda^k T^*M$, whose fiber vector space at $p \in M$ is $\Lambda^k(T_p^*M)$.

$V \qquad \Lambda^0 V = \mathbb{R}$

Definition A smooth k-form on a manifold M is a smooth section ω of $\Lambda^k(T_p^*M)$. We write $\Omega^k(M)$ for the vector space of smooth k-forms on M, i.e. $\Omega^k(M) := C^\infty(M, \Lambda^k T^*M)$.

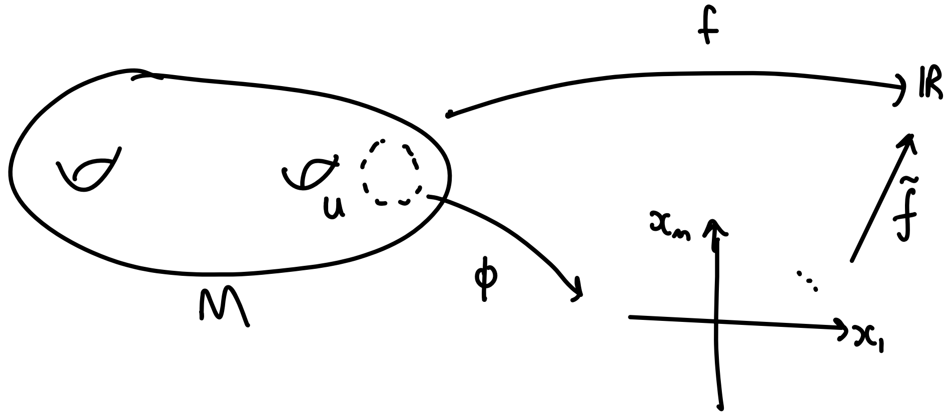
So, a k-form $\omega \in \Omega^k(M)$ consists of a smooth selection of a vector

$$\omega_p \in \Lambda^k(T_p^*M) \qquad p \in M$$

and is hence a measuring-stick on k-dimensional oriented volume elements in T_pM, for each $p \in M$. In particular, you can integrate $\omega \in \Omega^k(M)$ over a k-dimensional submanifold of M ... but we don't need this right now.

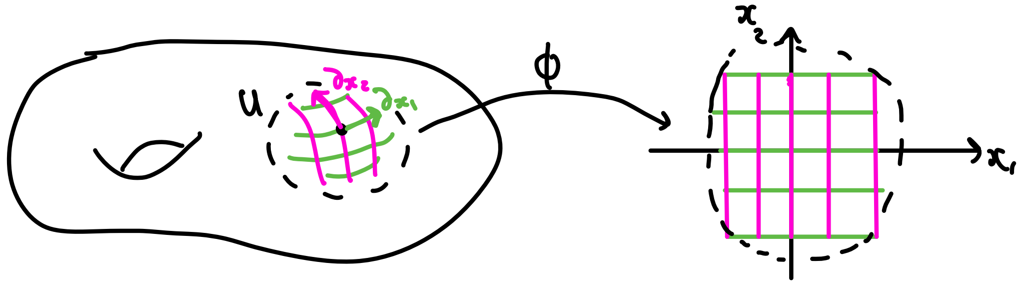
0-forms $\qquad \Omega^0(M) = C^\infty(M, \mathbb{R})$

A smooth function $f : M \longrightarrow \mathbb{R}$, when expressed in a local chart (U, ϕ) of M, becomes a function $\tilde{f}(x_1, \dots, x_m)$.



1-forms For all $p \in U$, we have the coordinate tangent vector basis

$$(\partial_{x_1})_p, \ldots, (\partial_{x_m})_p \in T_pM$$



So, at each $p \in M$, we have the dual basis

$$(dx_1)_p, \ldots, (dx_m)_p.$$ ← we could also write as $(\partial^{x_i})_p$.

So, locally on U, a 1-form can be written as

$$\omega = \omega_1(x_1, \ldots, x_m)\, dx_1 + \cdots + \omega_m(x_1, \ldots, x_m)\, dx_m$$

k-forms Similarly, for any coordinate chart

$$(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p \qquad 1 \leq i_1 < \cdots < i_k \leq m$$

is a basis for $\Lambda^k T_p^* M$, and so every $\omega \in \Omega^k(M)$ can be expanded locally in the coordinate chart as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \omega_{i_1 i_2 \cdots i_k}(x_1, \ldots, x_m)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

Example $M = \mathbb{R}^3$:

<u>0-form</u> $f = f(x,y,z)$

<u>1-form</u> $\alpha = \alpha_x(x,y,z)\,dx + \alpha_y(x,y,z)\,dy + \alpha_z(x,y,z)\,dz$

<u>2-form</u> $\beta = \beta_z(x,y,z)\,dy\wedge dz + \beta_y(x,y,z)\,dz\wedge dx + \beta_z(x,y,z)\,dx\wedge dy$

<u>3-form</u> $\omega = \omega(x,y,z)\,dx\wedge dy\wedge dz.$

Pullback of forms

If

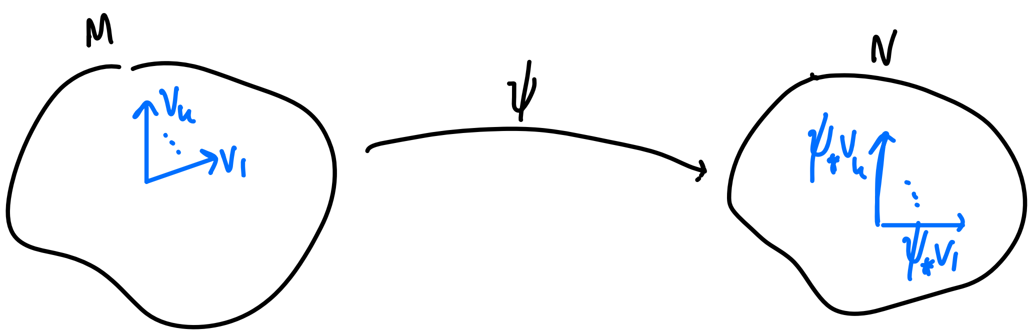
$$\psi : M \longrightarrow N$$

is a smooth map, then we get the <u>pullback</u> map

$$\psi^* : \Omega^k(N) \longrightarrow \Omega^k(M)$$

defined via:

$$\psi^*(\omega)(v_1\wedge\cdots\wedge v_k) = \omega(\psi_* v_1 \wedge\cdots\wedge \psi_* v_k)$$



1.7. The exterior derivative

There is a linear map

$$d : \Omega^0(M) \longrightarrow \Omega^1(M)$$

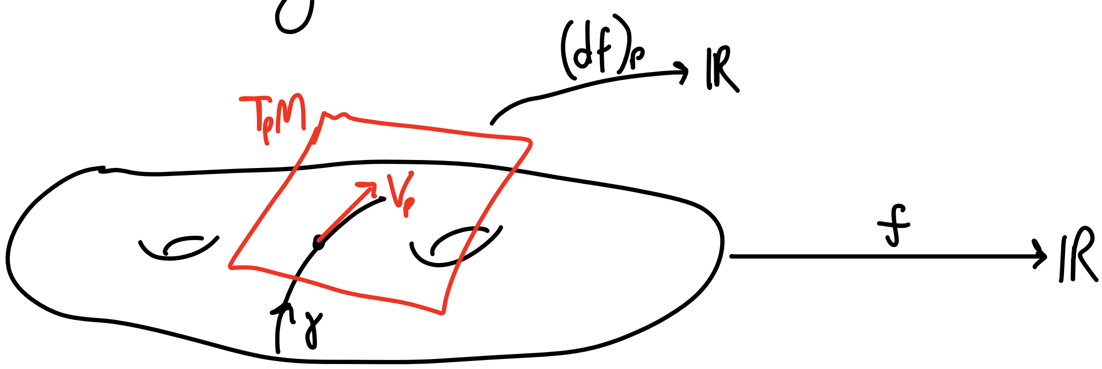
coordinate-free definition.

defined by

$$(df)_p(v_p) := \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))$$

df = "differential of f"

where γ is a curve representing $v \in T_pM$.



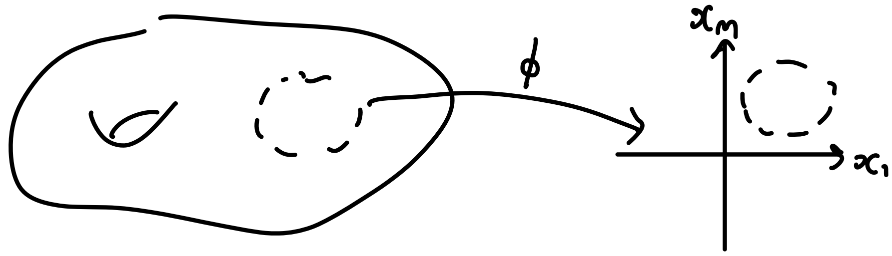
Exercise 1 Check $(df)_p$ is well-defined, i.e. only depends on the equivalence class of γ.

Locally, if $x_1, \dots, x_m$ are coordinates in a chart for M, then $f(p) = \tilde{f}(x_1, \dots, x_m)$, and

$$df = \frac{\partial \tilde{f}}{\partial x_1} dx_1 + \cdots + \frac{\partial \tilde{f}}{\partial x_m} dx_m$$

Exercise 2 Check this.

In particular, each coordinate x_i is a smooth function on $U \subseteq M$:



We calculate:

$$d(\text{the coordinate function } x_i) = \underbrace{dx_i}$$

the dual basis vector to ∂_{x_i}.

<u>Proof</u> $\quad d(\text{the coordinate function } x_i)\left(\partial_{x_j}\right)$

$$= \frac{d}{dt}\Big|_{t=0} x_i(\text{the path where we only change } x_j)$$

$$= \delta_{ij}. \qquad \square$$

We also check that the linear map

$$d : \Omega^0(M) \longrightarrow \Omega^1(M)$$

satisfies the "Leibniz rule":

$$d(fg) = f\,dg + g\,df.$$

<u>Proof</u> Locally,

$$d(fg) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}(\tilde{f}\tilde{g})\, dx_i$$

$$= \sum_{i=1}^{m}\left(\tilde{f}\frac{\partial \tilde{g}}{\partial x_i} + \tilde{g}\frac{\partial \tilde{f}}{\partial x_i}\right) dx_i$$

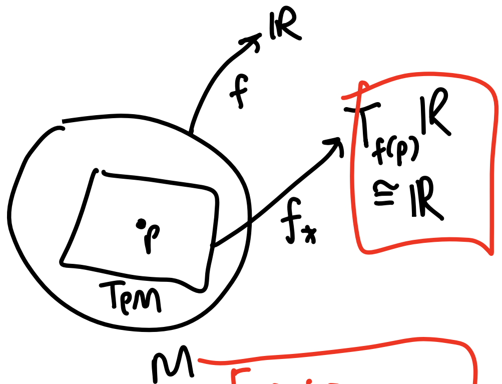
$$= \tilde{f} \sum_{i=1}^{m} \frac{\partial \tilde{g}}{\partial x_i} dx_i + \tilde{g} \sum_{i=1}^{m} \frac{\partial \tilde{f}}{\partial x_i} dx_i$$

$$= \tilde{f}\, dg + \tilde{g}\, df. \qquad \square$$

Also, if $\psi : M \to N$ is a smooth map, and $f \in \Omega^0(N)$, then

$$\psi^*(df) = d(\boxed{\psi^* f}) \qquad := f \circ \psi.$$

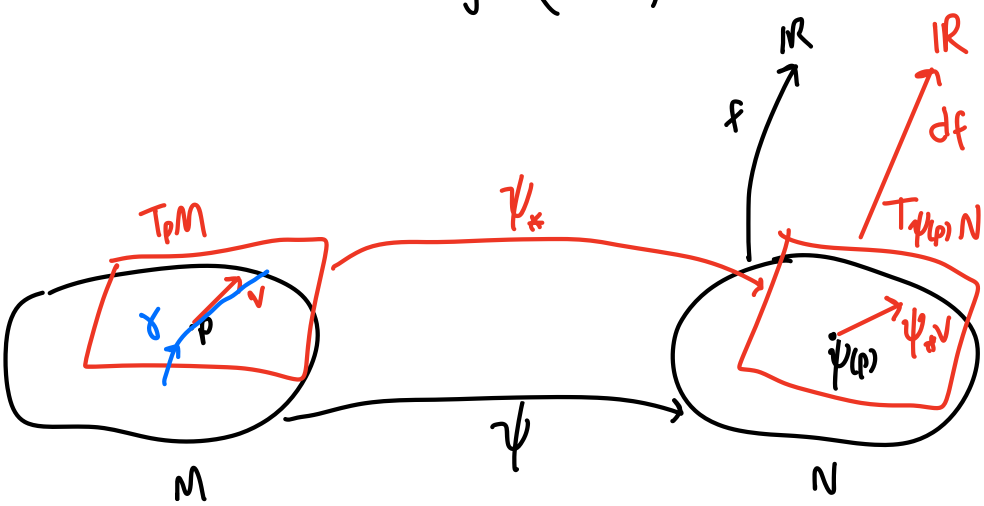
Check! At $p \in M$, let $v \in T_pM$.



$$LHS(v) = df(\psi_* v)$$
$$= f_*(\psi_* v)$$

Exercise

Check that $df(v) = f_*(v)$



$$RHS(v) = d(\psi^* f)(v)$$
$$= d(f \circ \psi)(v)$$
$$= (f \circ \psi)_*(v)$$
$$= f_*(\psi_* v) \qquad \text{(Chain rule).}$$

$\square$

We can extend d to a linear map ("exterior derivative")

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

as follows. Each k-form is locally a sum of forms of the form

$$\omega = f \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} .$$

We define

$$d\omega = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} \underbrace{dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}}_{k+1}.$$

Exercise Check that $d\omega$ does not depend on the coordinate charts used.

and extend the definition to all of $\Omega^k(M)$ by linearity.

Lemma $d^2 = 0$. (i.e. the composite $\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \Omega^{k+2}(M)$ is zero.)

Proof On a form of the form

$$\omega = f \, dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$

we have:

$$d\omega = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$

$$\therefore \quad d^2\omega = \sum_{i,j}^{m} \underbrace{\frac{\partial^2 f}{\partial x_i \partial x_j}}_{\text{symmetric by Clairaut's theorem (FTMC)}} \underbrace{dx_i \wedge dx_j}_{\text{antisymmetric}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$

$$= \sum_{i<j} \underbrace{\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right)}_{=0} dx_i \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$

$\square$

Example For $M = \mathbb{R}^3$, recall:

$$\underline{\text{0-form}} \quad f = f(x,y,z)$$

$$\underline{\text{1-form}} \quad \alpha = \alpha_x(x,y,z)\,dx + \alpha_y(x,y,z)\,dy + \alpha_z(x,y,z)\,dz \quad \nearrow \quad \vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$$

$$\underline{\text{2-form}} \quad \beta = \beta_x(x,y,z)\,dy\wedge dz + \beta_y(x,y,z)\,dz\wedge dx + \beta_z(x,y,z)\,dx\wedge dy \quad \to \quad \vec{\beta} = (\beta_x, \beta_y, \beta_z)$$

$$\underline{\text{3-form}} \quad \omega = \omega(x,y,z)\,dx\wedge dy\wedge dz \quad \to \quad \omega$$

So, eg.

$$\Omega^0 \xrightarrow{d} \Omega^1$$

$$f \longmapsto df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \nabla f = \operatorname{grad} f$$

$$\Omega^1 \xrightarrow{d} \Omega^2$$

$$\alpha \longmapsto d\alpha = \frac{\partial \alpha_x}{\partial x}\overbrace{dx\wedge dx}^{=0} + \frac{\partial \alpha_x}{\partial y}dy\wedge dx + \frac{\partial \alpha_x}{\partial z}dz\wedge dx$$

$$+ \frac{\partial \alpha_y}{\partial x}dx\wedge dy + \frac{\partial \alpha_y}{\partial y}\overbrace{dy\wedge dy}^{=0} + \frac{\partial \alpha_y}{\partial z}dz\wedge dy$$

$$+ \frac{\partial \alpha_z}{\partial x}dx\wedge dz + \frac{\partial \alpha_z}{\partial y}dy\wedge dz + 0$$

$$= \left(\frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y}\right)dx\wedge dy + \left(\frac{\partial \alpha_x}{\partial z} - \frac{\partial \alpha_z}{\partial x}\right)dz\wedge dx$$

$$+ \left(\frac{\partial \alpha_z}{\partial y} - \frac{\partial \alpha_y}{\partial z} \right) dy \wedge dz$$

$$\Omega^2 \xrightarrow{\ d\ } \Omega^3$$

$$\beta = \beta_z(x,y,z)\, dy \wedge dz + \beta_y(x,y,z)\, dz \wedge dx + \beta_z(x,y,z)\, dx \wedge dy$$

$$\beta \longmapsto d\beta = \left(\frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} + \frac{\partial \beta_z}{\partial z} \right) dx \wedge dy \wedge dz$$

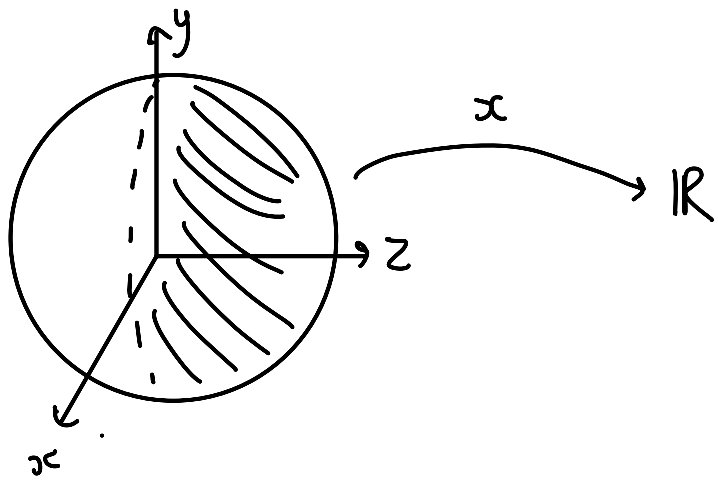
On $\mathbb{R}^3$, we can identify:

$$\begin{array}{ccc}
\Omega^0(\mathbb{R}^3) & & C^\infty(\mathbb{R}^3) \\
\downarrow d & & \downarrow \text{grad} \\
\Omega^1(\mathbb{R}^3) & & \mathrm{Vect}(\mathbb{R}^3) \\
\downarrow d & & \downarrow \text{curl} \\
\Omega^2(\mathbb{R}^3) & & \mathrm{Vect}(\mathbb{R}^3) \\
\downarrow d & & \downarrow \text{div} \\
\Omega^3(\mathbb{R}^3) & & C^\infty(\mathbb{R}^3)
\end{array}$$

$$\nabla \times \nabla f = \vec{0}$$

$$\nabla \cdot (\nabla \times \vec{V}) = \vec{0}$$

<u>Example</u> For $M = S^2$, consider $f(x) = x$



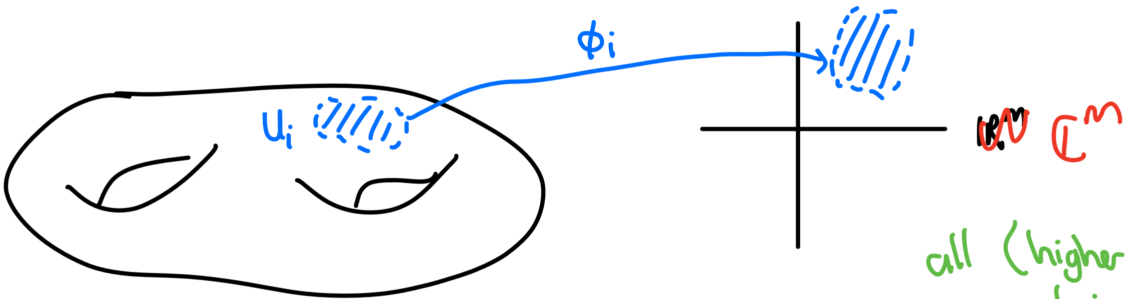
Since $x = \sin\theta\cos\phi$, in (x,y) coordinate system,

$$\begin{aligned} dx &= 1\,dx + 0.dy \\ &= 1.\left(\frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi\right) \\ &\qquad + 0.\left(\frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}d\phi\right) \\ &= \cos\theta\cos\phi\, d\theta - \sin\theta\sin\phi\, d\phi. \end{aligned}$$

2.1. Complex Manifolds and Holomorphic Maps

Definition An m-dimensional ~~smooth~~ holomorphic atlas $(U_i, \phi_i)_{i \in I}$ on a topological space M is an open cover (U_i) of M together with local charts (homeomorphisms)

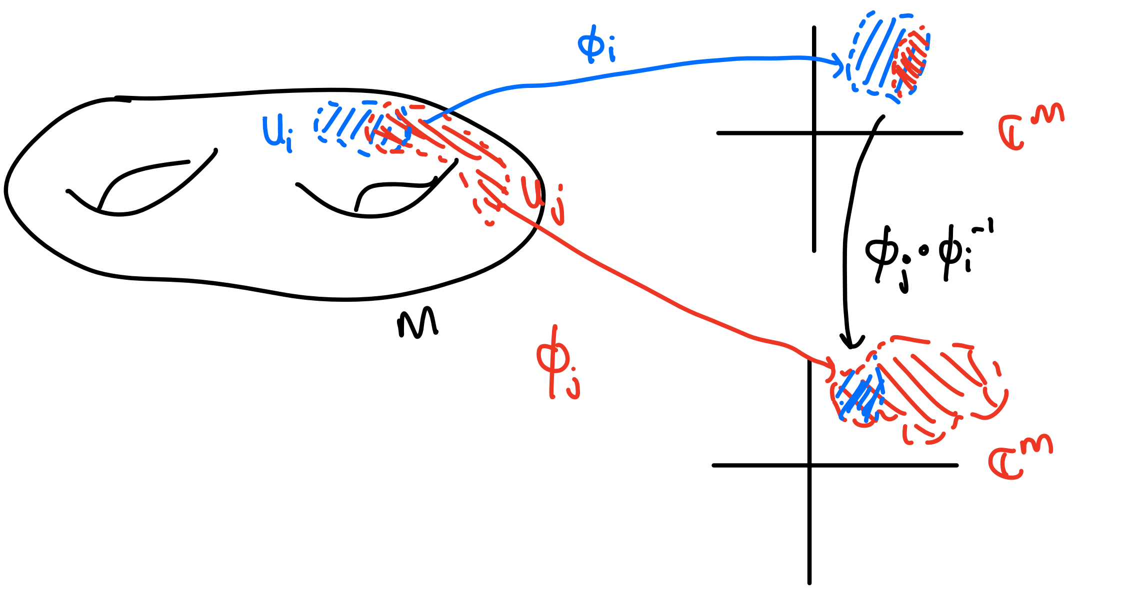
$$\phi_i : U_i \xrightarrow{\cong} \text{open subset of } \cancel{\mathbb{R}^m}\ \mathbb{C}^m$$



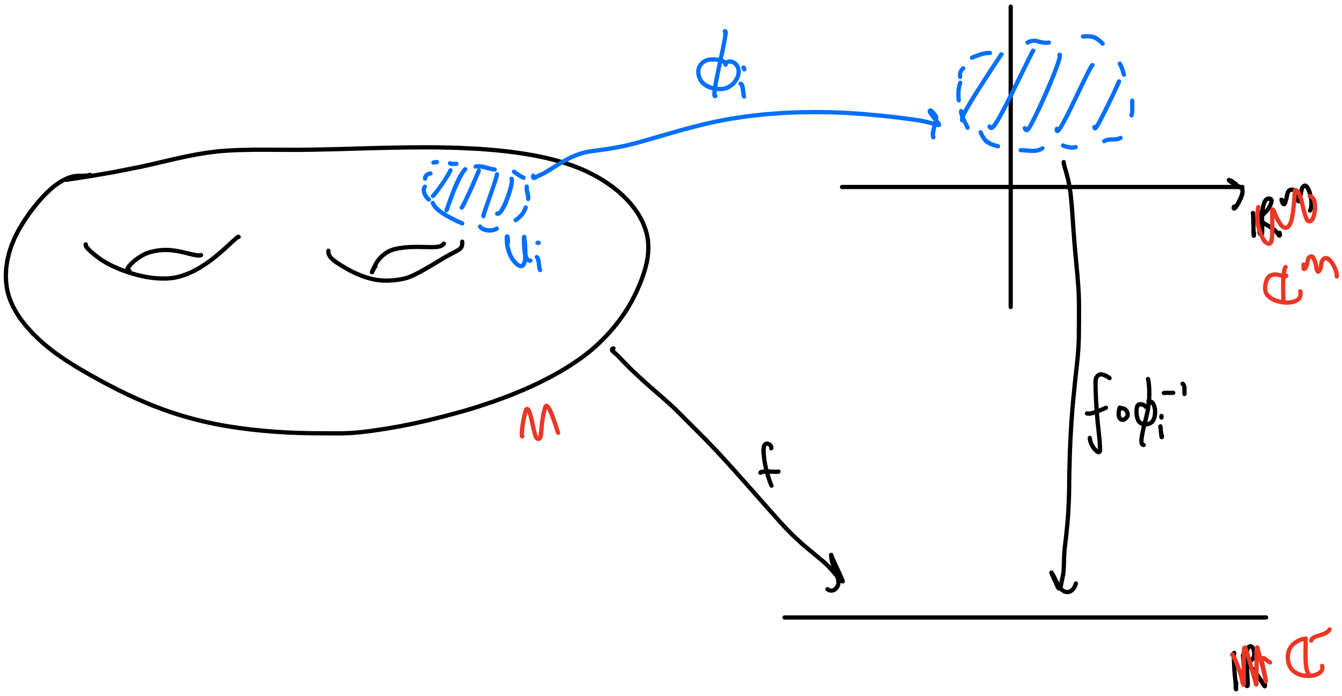
all (higher) partial derivatives exist and are holomorphic

satisfying, for all $U_i \cap U_j \neq \emptyset$,

$\phi_j \circ \phi_i^{-1}$ is a ~~smooth~~ holomorphic map (between open subsets of $\mathbb{C}^m$)



A holomorphic atlas on M allows us to say when a function $f: M \longrightarrow \mathbb{R}$ is ~~smooth~~ holomorphic: namely, we demand, for all charts (U_i, ϕ_i), that $f \circ \phi_i^{-1}$ is a ~~smooth~~ holomorphic map (from an open subset of ~~$\mathbb{R}^m$~~ $\mathbb{C}^m$ to ~~$\mathbb{R}$~~ $\mathbb{C}$, where we know what that means).



Definition Two ~~smooth~~ holomorphic atlases $(U_i, \phi_i)_{i \in I}$ and $(V_k, \psi_k)_{j \in J}$ on M are *equivalent* if they agree on which functions $f: M \longrightarrow$ ~~$\mathbb{R}$~~ $\mathbb{C}$ are ~~smooth~~ hol.

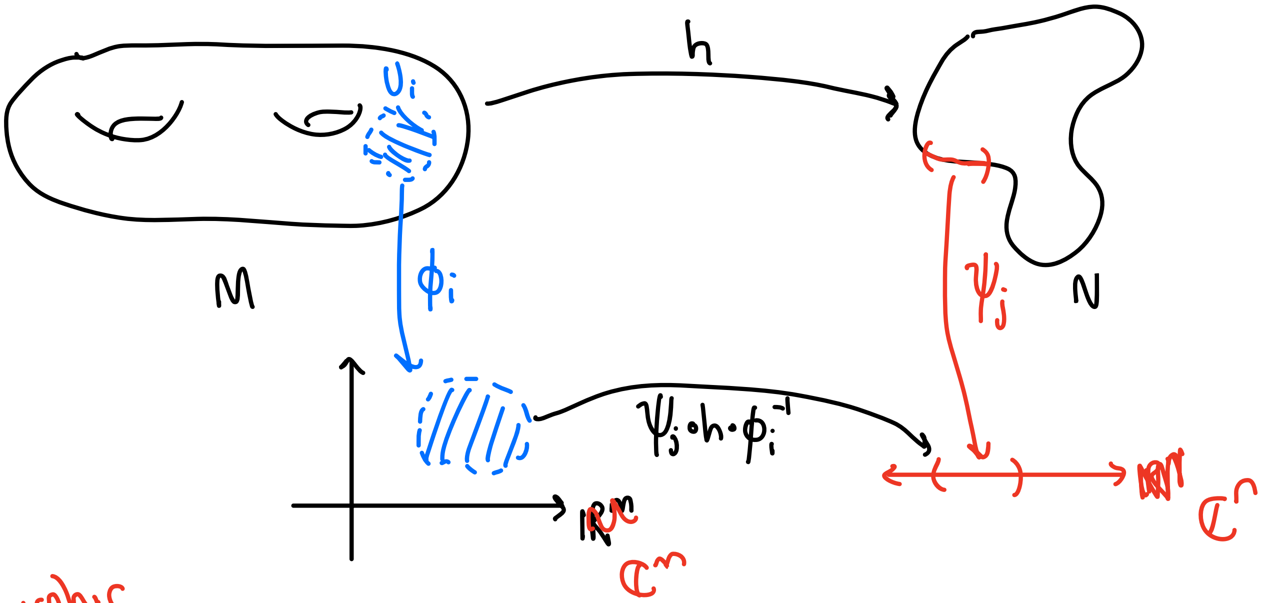
Definition An *m-dimensional ~~smooth~~ complex manifold* is a (Hausdorff, 2nd countable) topological space M equipped with an equivalence class of an m-dimensional ~~smooth~~ holomorphic atlas.

complex

Definition A map $h: M \longrightarrow N$ between ~~smooth~~ manifolds $(M, (U_i, \phi_i))$ and $(N, (V_j, \psi_j))$ is ~~smooth~~ holomorphic if, for all i, j,

$$\psi_j \circ h \circ \phi_i^{-1} : \text{open subset of } \cancel{\mathbb{R}^m}\ \mathbb{C}^n \longrightarrow \text{open subset of } \cancel{\mathbb{R}^n}\ \mathbb{C}^n$$

is smooth.



A ~~smooth~~ holomorphic map $h: M \longrightarrow N$ is called a holomorphic diffeomorphism if h^{-1} exists and is ~~smooth~~ hol.

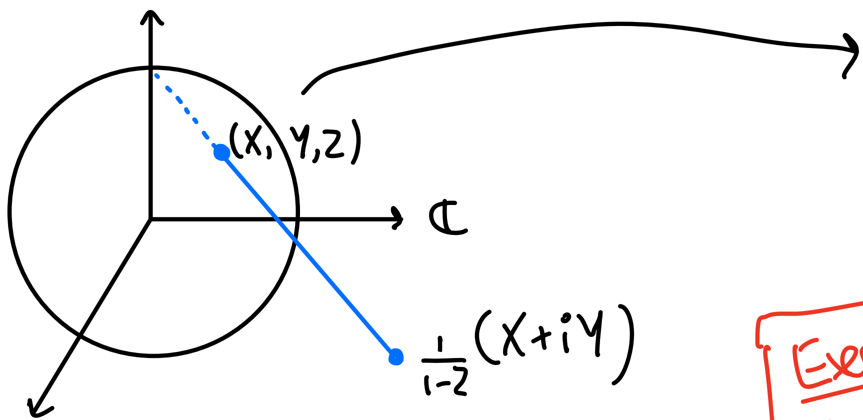
Example 1 $S^2 = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1 \}$ is a ~~smooth~~ complex manifold.

Hol. atlas: • $U_N = S^2 \setminus \{(0,0,1)\}$

$\phi_N : U_N \xrightarrow{\cong} \cancel{\mathbb{R}^2}\ \mathbb{C}$ stereographic projection from north pole

$(x,y,z) \longmapsto \frac{1}{1-z}(x - yi)$

$\frac{1}{1+x^2+y^2}(2x, 2y, x^2+y^2-1) \longleftarrow x + yi$



$\mathbb{C} \quad \mathbb{C}\setminus\{0\}$

$z \mapsto \frac{1}{z}$

$f(z)$

Exercise 1

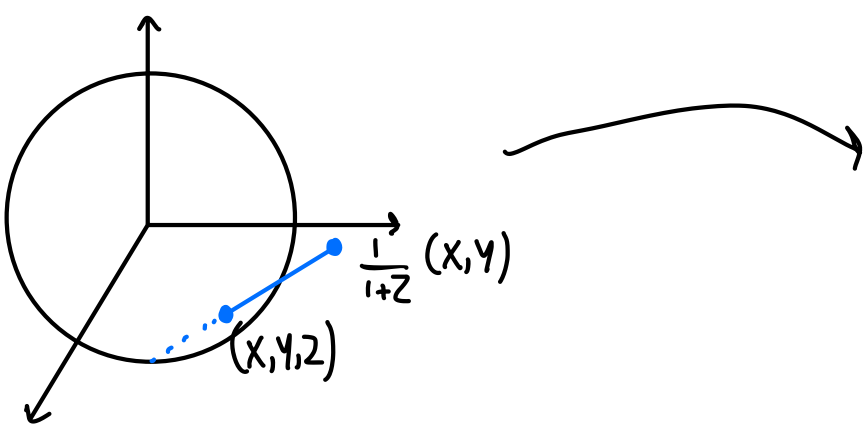
Check this!

• $U_S = S^2 \setminus \{0,0,-1\}$

$\phi_S : U_S \xrightarrow{\cong} \mathbb{R}^2$

$(x,y,z) \longmapsto \frac{x+iy}{1+z}$

$? \longleftarrow (x,y)$



$\mathbb{C} \quad \mathbb{C}\setminus\{0\}$

Example 2 $\mathbb{CP}^1 = \{\text{1-dim subspaces of } \mathbb{C}^2\}$.

$$= \frac{\mathbb{C}^2 \setminus \{(0,0)\}}{\sim}$$

$(z,w) \sim (\lambda z, \lambda w)$

where $[z:w] = [\lambda z : \lambda w]$ for all $\lambda \in \mathbb{C}^*$.

Can put charts on it as follows:

$$U_0 = \{ [z_0 : z_1] \in \mathbb{CP}^1 : z_0 \neq 0 \}$$

$$\begin{aligned} \phi_0 : U_0 &\xrightarrow{\cong} \mathbb{C} \\ [z_0 : z_1] &\longmapsto \frac{z_1}{z_0} \\ [1 : z] &\longleftarrow\!\shortmid\; z \end{aligned}$$

$$U_1 = \{ [z_0 : z_1] \in \mathbb{CP}^1 : z_1 \neq 0 \}$$

$$\begin{aligned} \phi_1 : U_1 &\xrightarrow{\cong} \mathbb{C} \\ [z_0 : z_1] &\longmapsto \frac{z_0}{z_1} \\ [z : 1] &\longleftarrow\!\shortmid\; z \end{aligned}$$

Transition functions:

$$z \overset{\phi_0^{-1}}{\longmapsto} [1 : z] \overset{\phi_1}{\longmapsto} \frac{1}{z}$$

which is holomorphic on $\phi_0(U_0 \cap U_1) = \mathbb{C}^*$.

Indeed, we have a holomorphic diffeomorphism

$$\begin{tikzcd}
\psi : S^2 \arrow[rr] \arrow[dr, "\substack{\text{stereographic}\\ \text{projection}\\ \text{from South pole}}"'] & & \mathbb{CP}^1 \\
& \mathbb{C} \cup \{\infty\} \arrow[ur] &
\end{tikzcd}
\qquad
\begin{matrix} [z:1] & [1:0] \\ \uparrow & \uparrow \\ z & \infty \end{matrix}$$

Exercise 2 Check this.

More generally,

$$\mathbb{CP}^n = \{ \text{1-dimensional subspaces of } \mathbb{C}^{n+1} \}$$

can be equipped with a holomorphic atlas in a similar way.

Exercise 3. Supply the details.

A <u>Riemann surface</u> is a 1-dimensional complex manifold.

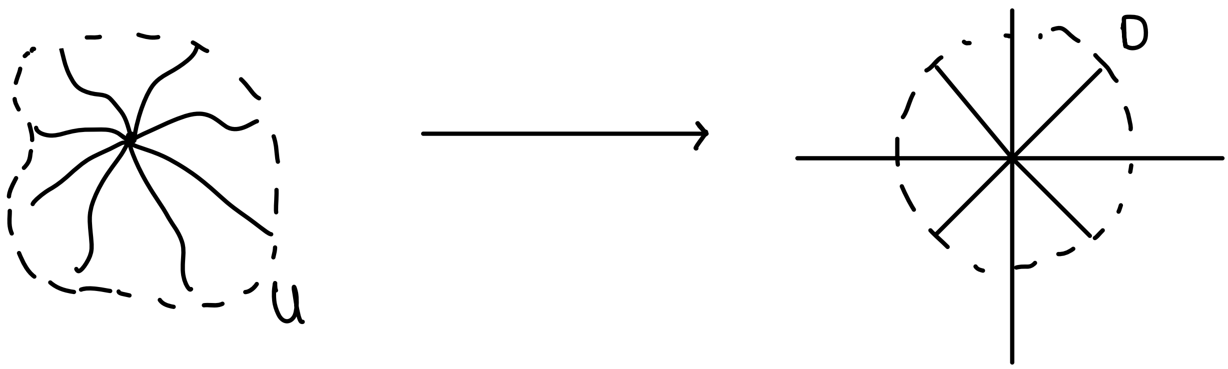
Example 3

- Every open set in $\mathbb{C}$ is a complex manifold, eg.

$$\mathbb{C}, \quad D = \{ z : |z| < 1 \}$$

Recall the Riemann mapping theorem:

$$\boxed{\text{Every connected open subset } U \subseteq \mathbb{C} \text{ which is not all of } \mathbb{C} \text{ is holomorphically in bijective correspondence with } D.}$$



But, $\mathbb{C} \not\cong D$, because

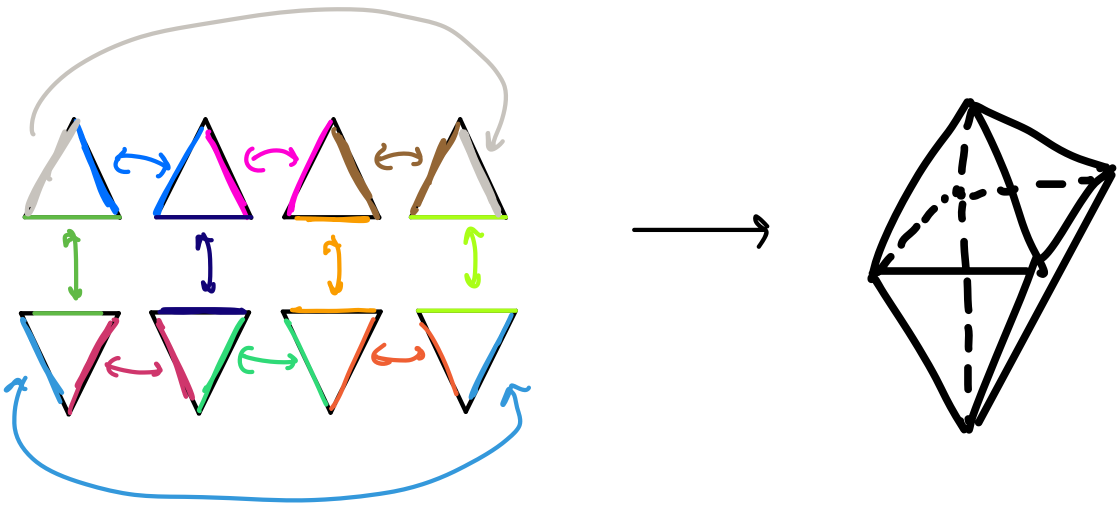
3-dim Lie group $\longrightarrow$ $\mathrm{Aut}(D) = \left\{ z \longmapsto \frac{az+b}{\bar{b}z+\bar{a}}, \quad |a|^2 - |b|^2 = 1 \right\}$

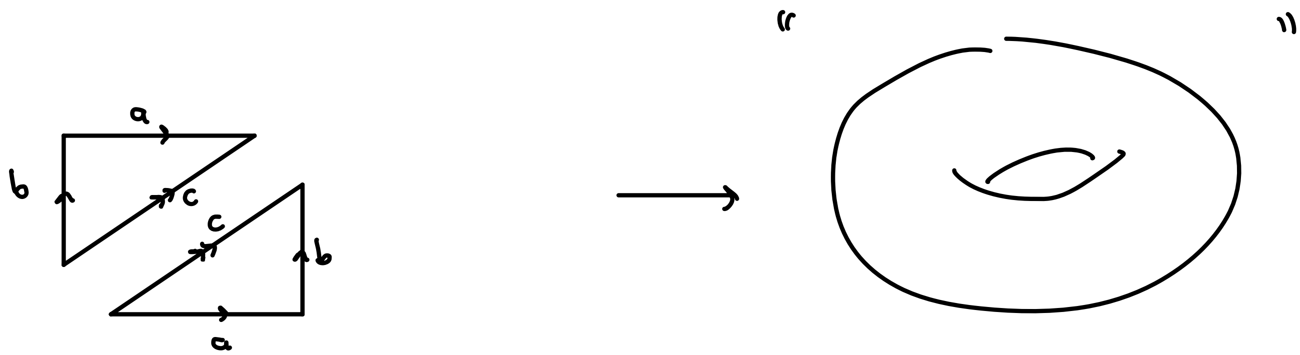
while

4-dim Lie group $\longrightarrow$ $\mathrm{Aut}(\mathbb{C}) = \{ z \longmapsto az+b, \quad a \in \mathbb{C}^*, b \in \mathbb{C} \}$

Example 4 Suppose you have a topological surface obtained by gluing together the edges of a collection of Euclidean triangles together in pairs:

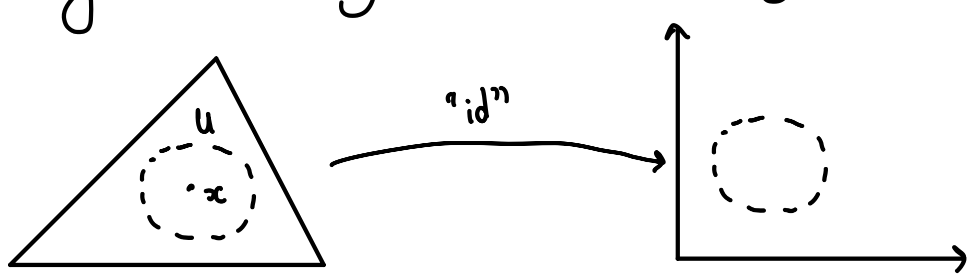
eg.:

- 

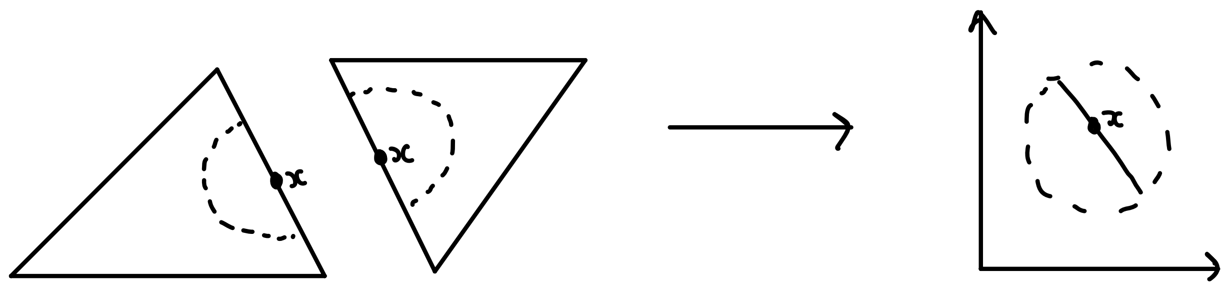
- 

Then you can place a holomorphic atlas on it as follows. Let x be a point on the surface.

- If x is not a vertex or on an edge, choose a disk U around x lying entirely in the triangle, and use the "identity" chart:



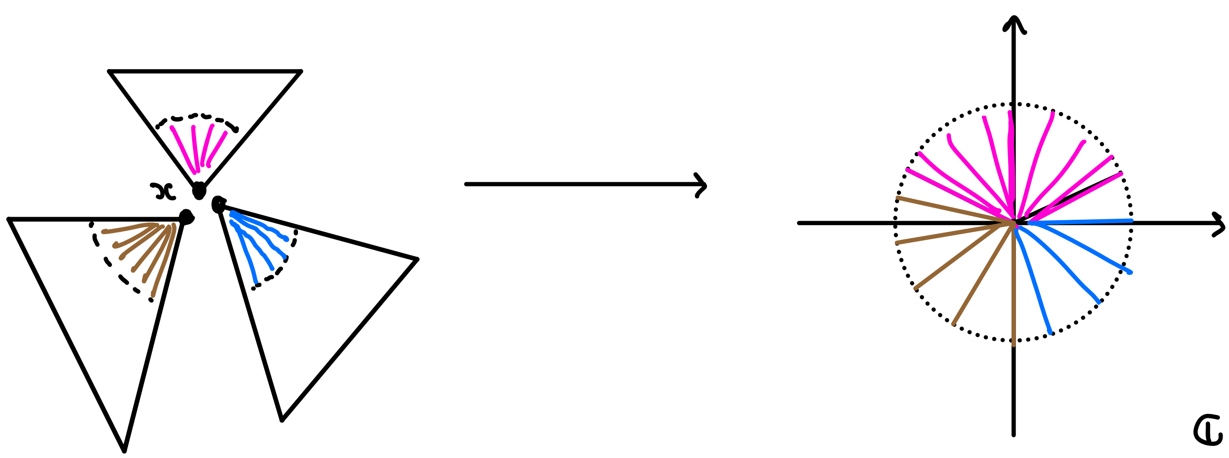
- If x lies on an edge e (but is not a vertex), then there are two triangles having edge e. The chart is the disk obtained as two half-disks joined together:



- If x is a vertex, then consider the disk U made up of sectors composed out of all the triangles incident on x: They span an angle α (which might not equal 2π). Map

$$z \longmapsto z^{\frac{2\pi}{\alpha}}$$

so as to map U 1-1 onto a disk in $\mathbb{C}$:



$\boxed{\text{Exercise 3 Check that this is a holomorphic atlas!}}$

Example 5 (Algebraic curves) Suppose we are given a polynomial $p(z,w)$.
Set

$$X = \{(z,w) \in \mathbb{C}^2 : p(z,w) = 0\}$$

Suppose that for all $(z,w) \in X$, $\nabla p = (p_z, p_w) \neq (0,0)$. Note what this condition means: if $p_w \neq 0$ at (z_0, w_0), then near (z_0,w_0) we can express $w(z)$ in a holomorphic way (by the holomorphic version of the Implicit Function Theorem earlier).

So, we can parametrize the points of X using z as a local coordinate:

$$z \longmapsto (z, w(z))$$

Similarly, if $p_w \neq 0$ at (z_0, w_0), then we can locally express $z(w)$, and get a local parametrization

$$w \longmapsto (z(w), w).$$

Exercise!

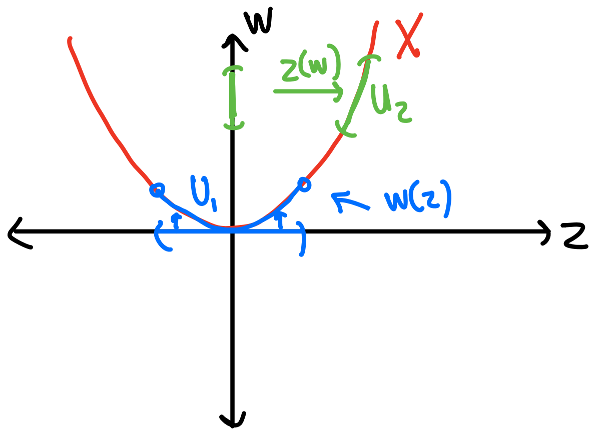
In this way we get a holomorphic atlas for X. We call X a <u>smooth algebraic curve</u>.

For example,

$$p(z,w) = w - z^2 \qquad X = \{(z,w) : w - z^2 = 0\}$$

$$p_w = 1$$
$$p_z = -2z$$

Picture of X in $\mathbb{R}^2$:



(More generally, in higher dimensions, a space of the form

$$X = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : p_i(z_1, \cdots, z_n) = 0\}$$

for some polynomial p_i where rank $\left(\frac{\partial p_i}{\partial z_j}\right)$ is maximal on X has a natural holomorphic atlas ... we call it a <u>smooth algebraic variety</u>).

The way to understand this is that the Riemann surface X comes with a projection map onto each coordinate, e.g.

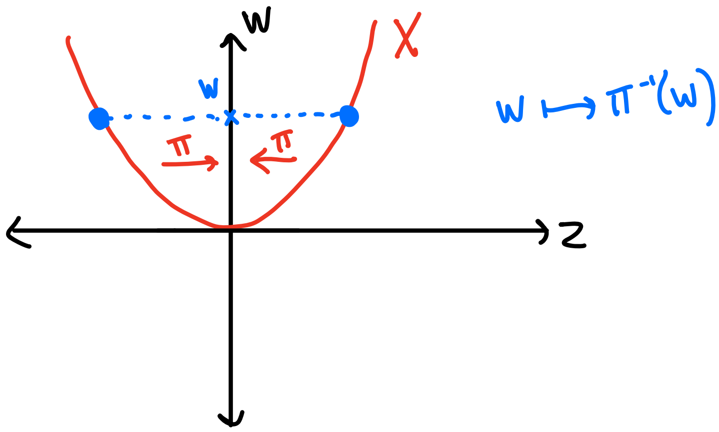
$$\pi : X \longrightarrow \mathbb{C}$$
$$(z,w) \longmapsto w$$

And so we think of X as a rigorous, holomorphic construct which expresses z as a <u>multivalued</u> function of w:

$$w \longmapsto \pi^{-1}(w)$$

In our example, this makes rigorous the function

$$\text{“} \quad w \longmapsto \sqrt{w} \quad \text{”} .$$



This $X \subseteq \mathbb{C}^2$ is not a <u>compact</u> Riemann surface. The compact version $\overline{X} \subseteq \mathbb{CP}^2$ is obtained by homogenizing $p(z,w)$ to a <u>homogenous</u> polynomial $P(t,z,w)$ by adding powers of t, eg

$$p(z,w) = z^5 - 2z^2 w - z^3 w^2 + w$$

$$\rightsquigarrow \quad P(t\ z,w) = z^5 - 2t^2 z^2 w - z^3 w^2 + t^4 w$$

and then setting

$$\overline{X} := \left\{ (t:z:w) \in \mathbb{CP}^2 \ : \ P(t,z,w) = 0 \right\}$$

will give us a compact Riemann surface [providing $\nabla P \neq 0$ on $\overline{X}$], a <u>smooth projective algebraic curve</u>.

Note that

$$\overline{X} = X \cup \underbrace{\{ (0:z:w) \in \mathbb{CP}^2 : P(0,z,w) = 0 \}}_{\text{finite number of points "at infinity"}}$$

In our example, the points at infinity are given by:

$$(0:z:w) \quad : \quad z^5 - z^3 w^2 = 0$$

So we get three points at infinity:

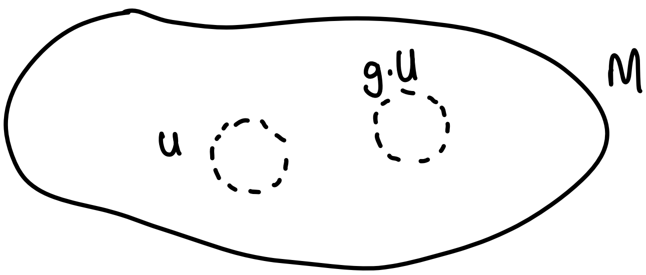
$$(0:1:\overset{\pm 1}{w}) \quad : \quad 1 - w^2 = 0 \quad \Rightarrow \quad w = \pm 1$$

$$(0:0:1) \quad : \quad 0 = 0 \quad \checkmark$$

__Example 6__ __Quotient spaces__ If M is a complex manifold, and G is a group which acts properly discontinuously on M, then M/G is a Riemann surface (by inheriting the charts from M).

i.e. every $p \in M$ has a neighborhood U such that

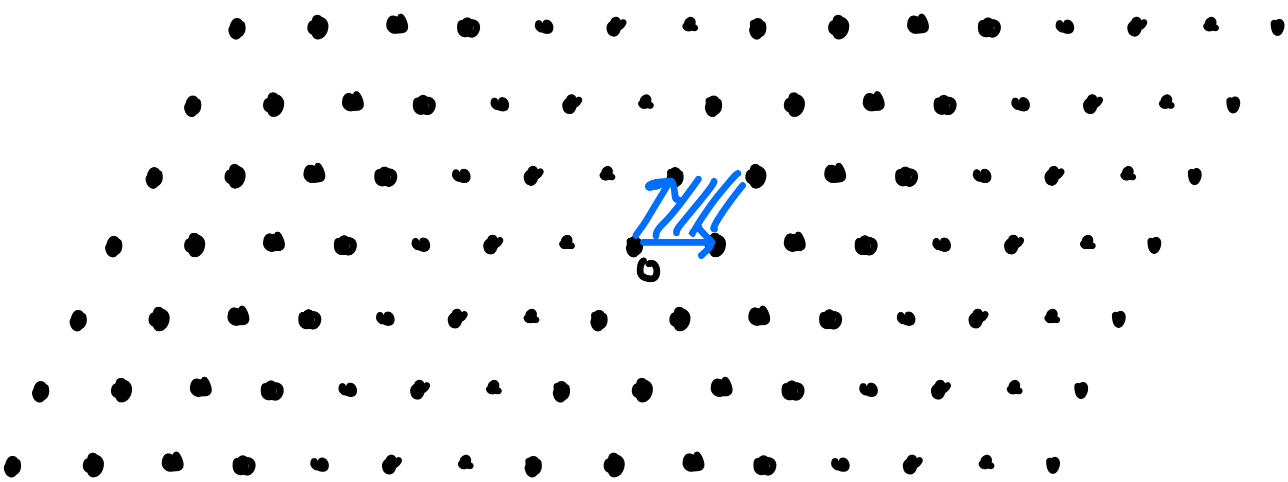
$$g\cdot U \cap U \text{ is empty} \iff g=e$$



eg. $\Lambda \leq \mathbb{C}$ a __lattice__ (a discrete subgroup of $\mathbb{C}$). Then

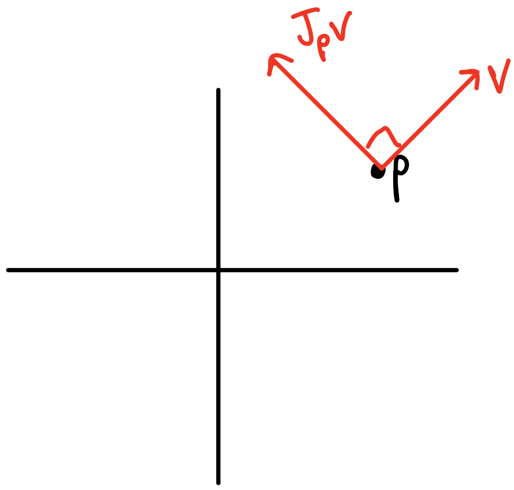
$$X = \mathbb{C}/\Lambda$$

is a Riemann surface.



2.2. Almost Complex Structures

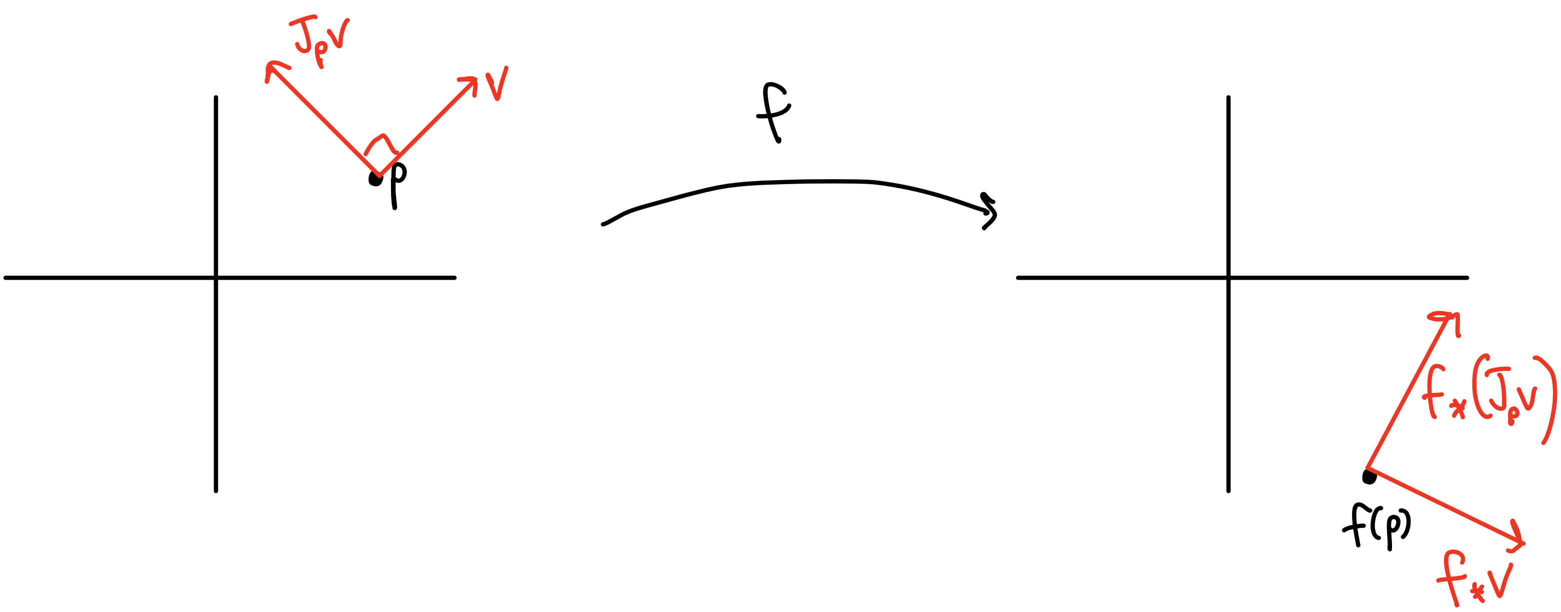
At every $p \in \mathbb{R}^2$, we have the "rotate counterclockwise by $90°$" map $J_p : T_p\mathbb{R}^2 \longrightarrow T_p\mathbb{R}^2$:



Lemma A smooth map $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is holomorphic (when thought of as a map $f: \mathbb{C} \to \mathbb{C}$) if and only if

$$f_*(J_p v) = J_{f(p)}(f_* v)$$

for all $p \in \mathbb{R}^2$.



<u>Proof</u> It is sufficient to check this on the basis

$$\partial_x \, , \, \partial_y$$

for $T_p \mathbb{R}^2$. Note that

$$J(\partial_x) = \partial_y \, , \qquad J(\partial_y) = -\partial_x$$

Write

$$f(x,y) = \big(u(x,y), \, v(x,y) \big).$$

Then:

$$f_*(J\partial_x) = f_*(\partial_y)$$

$$= \frac{\partial u}{\partial y} \partial_x + \frac{\partial v}{\partial y} \partial_y$$

while

$$J(f_* \partial_x) = J\left(\frac{\partial u}{\partial x} \partial_x + \frac{\partial v}{\partial x} \partial_y \right)$$

$$= \frac{\partial u}{\partial x} \partial_y - \frac{\partial v}{\partial x} \partial_x$$

So

$$f_*(J\partial_x) \overset{①}{=} J(f_* \partial_x) \iff \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}}$$

Similarly,

$$f_*(J\partial_y) \overset{2}{=} J(f_*\partial_y)$$

yields the exact same set of equations :

$$\begin{aligned} f_*(J\partial_y) &= f_*(-\partial_x) \\ &= -f_*(\partial_x) \\ &= -\left[\frac{\partial u}{\partial x}\partial_x + \frac{\partial v}{\partial x}\partial_y\right] \end{aligned}$$

$$\begin{aligned} J(f_*\partial_y) &= J\left[\frac{\partial u}{\partial y}\partial_x + \frac{\partial v}{\partial y}\partial_y\right] \\ &= \frac{\partial u}{\partial y}\partial_y - \frac{\partial v}{\partial y}\partial_x \end{aligned}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

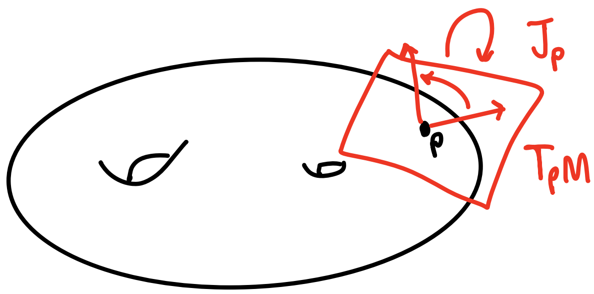
These are precisely the Cauchy-Riemann equations, which are the definition of

$$f(x,y) = u(x,y) + iv(x,y)$$

to be holomorphic !

$\square$

Definition An almost complex structure on a smooth real manifold M is a smooth section J of $\underbrace{\mathrm{End}(TM)}_{\cong T^*M \otimes TM}$ satisfying $J_p^2 = -\mathrm{id}_p$ on each tangent space. An almost complex manifold is a manifold equipped with an almost complex structure.



Example • $\mathbb{R}^2$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on each tangent space, i.e.

$$J(\partial_x) = \partial_y, \qquad J(\partial_y) = -\partial_x$$

• $\mathbb{R}^{2n}$, where:

$$J(\partial_{x_i}) = \partial_{y_i} \qquad J(\partial_{y_i}) = -\partial_{x_i}$$

• Any complex manifold M has an almost complex structure. Let $p \in M$. Choose a holomorphic chart $(z_1, \dots, z_m)$ around p. Then

$$(x_1, y_1, x_2, y_2, \dots, x_m, y_m)$$

are real coordinates around p, and we set

$$J(\partial_{x_i}) = \partial_{y_i} \quad , \quad J(\partial_{y_i}) = -\partial_{x_i} .$$

<u>Exercise 1</u> Check that this definition does not depend on the holomorphic chart used.

The converse is not true in general! Not every almost complex manifold can be equipped with a holomorphic atlas.

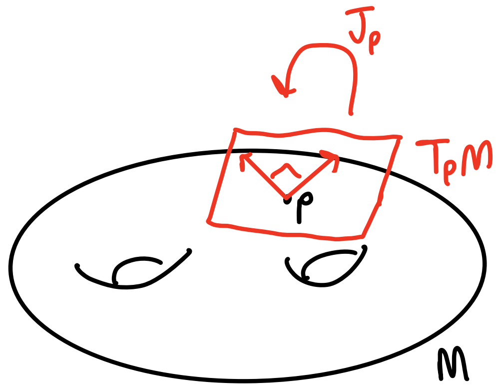
But : much of the theory of complex manifolds only needs the underlying almost complex structure, and not the holomorphic charts. For instance, the definition of a holomorphic map!

<u>Lemma</u> A smooth map $f : M \longrightarrow N$ between complex manifolds is holomorphic if and only if f_* preserves the almost complex structure, i.e.

$$f_* J_p = J_{f(p)} f_* \qquad \forall\, p \in M.$$

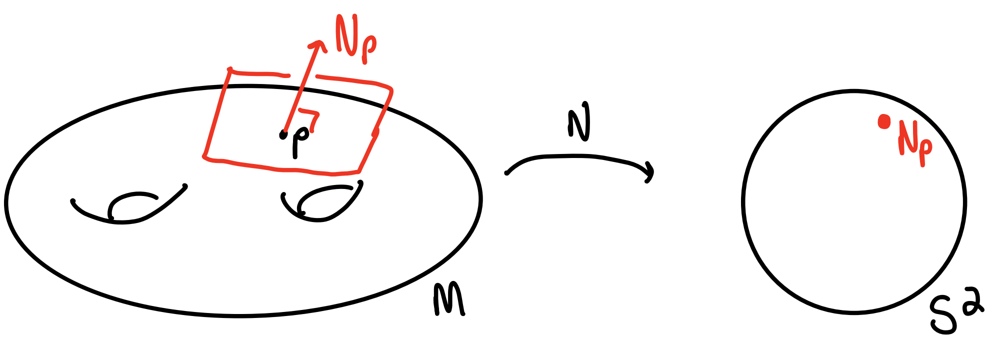
<u>Exercise 2</u> Prove this!

<u>Example</u> Suppose $M \subseteq \mathbb{R}^3$ is an oriented smooth surface embedded in $\mathbb{R}^3$. Then M inherits an almost-complex structure J, by rotating counterclockwise by 90° in each tangent space:



What holomorphic chart is compatible with J? Well, we have the <u>Gauss map</u>

$$N : M \longrightarrow S^2$$
$$p \longmapsto \text{outward unit normal vector at } p$$



It turns out that N is holomorphic (as a map between almost complex manifolds) if (but not <u>only</u> if) M is a <u>minimal surface</u>, i.e. the eigenvalues of the shape operator

$$S_p : T_pM \longrightarrow T_pM$$
$$v \longmapsto \text{derivative of } N \text{ in the direction of } v$$

are $\pm K$ (i.e. the principal curvatures are opposite).

2.3. Some almost-complex linear algebra

Every real vector space V has a complexification

$$V_{\mathbb{C}} := \{ \text{formal combinations } v_1 + i v_2 \quad , \; v_1, v_2 \in V \}$$

which is a complex vector space.

Note that we can also write

$$V_{\mathbb{C}} = V \underset{\mathbb{R}}{\otimes} \mathbb{C}$$

via the identification, for $v \in V$,

$$v \longmapsto v \otimes 1$$
$$iv \longmapsto v \otimes i$$

If

$$e_1, \dots, e_m$$

is a basis for the real vector space V

$$e_1, \dots, e_m$$

is a basis for the complex vector space $V_{\mathbb{C}}$

So: V is an m-dimensional real vector space

$V_{\mathbb{C}}$ is an m-dimensional complex vector space

$V_{\mathbb{C}}$ is a $2m$-dimensional real vector space

Moreover, any linear map

$$A : V \longrightarrow W$$

between real vector spaces extends to a complex-linear map

$$A_{\mathbb{C}} : V_{\mathbb{C}} \longrightarrow W_{\mathbb{C}}$$

$$v_1 + i v_2 \longmapsto A v_1 + i A v_2$$

<u>Exercise 1</u> Check that $A_{\mathbb{C}}$ is a complex-linear map.

Suppose we have a real finite-dimensional vector space V and a linear map

$$J : V \longrightarrow V, \qquad J^2 = -\mathrm{id}$$

<u>Lemma</u> We can find a basis for V of the form

$$e_1, f_1, e_2, f_2, \cdots, e_n, f_n$$

where $\quad Je_i = f_i, \quad Jf_i = -e_i.$

<u>Exercise 2</u> : Prove this!

In particular, this means the dimension of V must be <u>even</u>.

Now, $J^2 = -\mathrm{id} \Rightarrow$ eigenvalues of J are $\pm i$. So, its eigenvectors don't live in V, but rather in $V_{\mathbb{C}}$. In other words, we extend J to a complex-linear map

$$\begin{aligned} J_{\mathbb{C}} : V_{\mathbb{C}} &\longrightarrow V_{\mathbb{C}} \\ v_1 + i v_2 &\longmapsto J v_1 + i J v_2 \end{aligned}$$

and then we can decompose $V_{\mathbb{C}}$ into the eigenspaces of $J_{\mathbb{C}}$.

Let's calculate these.

$$J(v_1 + i v_2) = i(v_1 + i v_2)$$

$$\iff \quad Jv_1 + iJv_2 = -v_2 + iv_1$$

$$\iff \quad v_2 = -Jv_1$$

So, $V^{1,0} := \mathrm{Eig}_{\lambda=i} = \{ v - iJv \; : \; v \in V \} \subseteq V_{\mathbb{C}}$

and $V^{0,1} := \mathrm{Eig}_{\lambda=-i} = \{ v + iJv \; : \; v \in V \} \subseteq V_{\mathbb{C}}$

$V \quad e_1, f_1, \dots, e_n, f_n \qquad Je_i = f_i$

$V_{\mathbb{C}} \quad e_1, f_1, \dots, e_n, f_n \qquad Jf_i = -e_i$

and we have decomposed:

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

$$= \underbrace{\{ v - iJv : v \in V \}}_{\substack{\text{basis } a_i := e_i - if_i \\ i = 1 \dots m}} \oplus \underbrace{\{ v + iJv : v \in V \}}_{\substack{\text{basis } \bar{a}_i := e_i + if_i \\ i = 1 \dots m}}$$

Note that we have an antilinear bijection:

$$V^{1,0} \longrightarrow V^{0,1}$$

$$w \longmapsto \bar{w}$$

Oh, so:

- Given a real $2n$-dimensional vector space V, we have its complexification $V_{\mathbb{C}}$, which is a $2n$-dimensional complex vector space.

- Given a real $2n$-dimensional vector space V and a linear map

$$J : V \longrightarrow V, \qquad J^2 = -id$$

we can decompose $V_{\mathbb{C}}$ into the $\pm i$ eigenspaces of $J_{\mathbb{C}}$

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

- But given (V, J), we can also regard the $2n$-dimensional real vector space V as an n-dimensional complex vector space, by defining scalar multiplication by complex numbers via

$$i \cdot v := Jv$$

Keep this in mind!

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2.4. Decomposition of forms

Given an almost-complex manifold (M, J), we can decompose the complexification of each tangent space into the $\pm i$ eigenspaces of $J_{\mathbb{C}}$:

$$T_pM \otimes \mathbb{C} = T_p^{1,0}M \oplus T_p^{0,1}M$$

So, the <u>complexified tangent bundle</u> naturally splits as

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M.$$

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

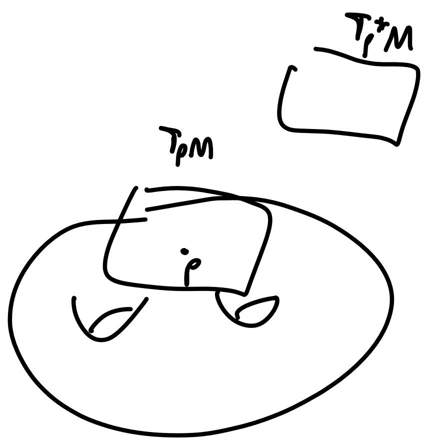
$$V_{\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$$

$$= \underbrace{(V_{\mathbb{C}}^*)^{1,0}}_{= \{ f : V_{\mathbb{C}} \to \mathbb{C} \;:\; f(v) = 0 \text{ for } v \in V^{0,1} \}} \oplus (V_{\mathbb{C}}^*)^{0,1}$$

$$\Lambda^k V_{\mathbb{C}}^* = \bigoplus_{\substack{a,b \\ a+b=k}} \underbrace{\Lambda^{a,0} V^* \otimes \Lambda^{0,b} V^*}_{\Lambda^{a,b} V^*}$$

$$V = A \oplus B$$

$$\Lambda^k V = \bigoplus_{a+b=k} \Lambda^a A \otimes \Lambda^b B$$



(M, J) almost complex manifold

$$T_p M \circlearrowleft J_p \quad , \quad J_p^2 = -\mathrm{id}$$

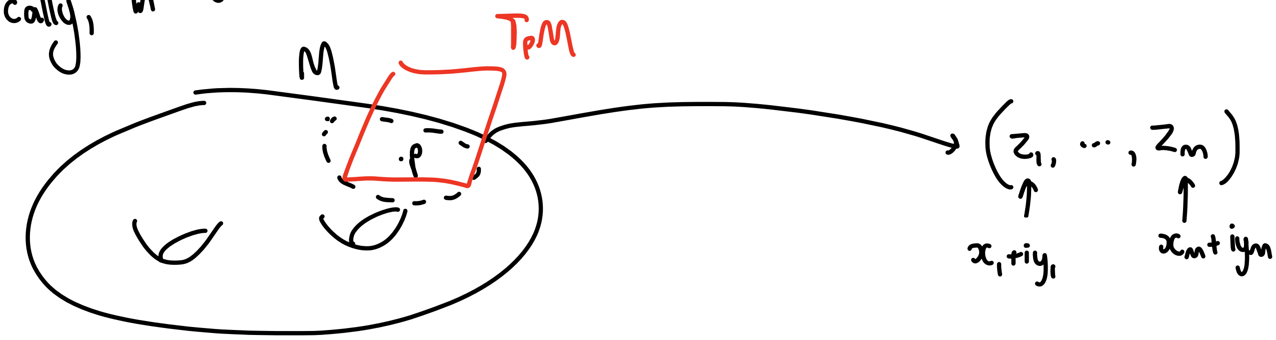
$$T_p^* M \otimes \mathbb{C} = (T_p^* M)^{1,0} \oplus (T_p^* M)^{0,1}$$

$$\Lambda^k (T_p^* M \otimes \mathbb{C}) = \bigoplus_{a+b=k} \Lambda^{a,b} (T_p^* M)$$

$\therefore$ Holds on tangent bundle, and hence on sections

$$\boxed{\Omega^k(M)\otimes\mathbb{C} = \bigoplus_{a+b=k}\Omega^{a,b}(M)}$$

Locally, in a chart:



$\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_m}, \partial_{y_m}$ basis for T_pM

$$J\partial_{x_i} = \partial_{y_i}, \quad J\partial_{y_i} = -\partial_{x_i}$$

$$\partial_{z_i} := \tfrac{1}{2}(\partial_{x_i} - i\partial_{y_i}) \in T_p^{1,0}M \subseteq T_pM\otimes\mathbb{C}$$

$$\partial_{\bar{z}_i} := \tfrac{1}{2}(\partial_{x_i} + i\partial_{y_i}) \in T_p^{0,1}M \subseteq T_pM\otimes\mathbb{C}$$

$$J\partial_{z_i} = i\partial_{z_i}, \quad J\partial_{\bar{z}_i} = -i\partial_{z_i}$$

Basis for $T_pM\otimes\mathbb{C} = \partial_{z_1}, \dots, \partial_{z_m}, \partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_m}$

$\therefore$ Have a dual basis for $T_p^*M\otimes\mathbb{C} = \underbrace{dz_1, \dots, dz_m}_{\text{basis for } (T_p^*M)^{1,0}}, \underbrace{d\bar{z}_1, \dots, d\bar{z}_m}_{\text{basis for } (T_p^*M)^{0,1}}$

Notice:

$$dz_i = dx_i + i\,dy_i \qquad \text{since } (dx_i + i\,dy_i)\left(\tfrac{1}{2}(\partial_{x_i} - i\partial_{y_i})\right)$$
$$= \tfrac{1}{2}(1+1) = 1.$$

$$d\bar{z}_i = dx_i - i\,dy_i$$

$\therefore$ Basis for $\Lambda^2(T_p^*M \otimes \mathbb{C}) = \underbrace{\Lambda^{2,0}(T_p^*M)}_{dz_i \wedge dz_j} \oplus \underbrace{\Lambda^{1,1}(T_p^*M)}_{dz_i \wedge d\bar{z}_j} \oplus \underbrace{\Lambda^{0,2}(T_p^*M)}_{d\bar{z}_i \wedge d\bar{z}_j}$

eg. M is 4 real-dim (2 complex dim).

$$\underset{\text{basis}}{(T_p^*M \otimes \mathbb{C})} = \underbrace{(T_p^*M)^{1,0}}_{dz_1,\, dz_2} \oplus \underbrace{(T_p^*M)^{0,1}}_{d\bar{z}_1,\, d\bar{z}_2}$$

$$\boxed{dz_i := dx_i + i dy_i}$$

6

$$\boxed{\underset{\text{basis}}{\Lambda^2(T_p^*M \otimes \mathbb{C})}} = \underbrace{\Lambda^{2,0}}_{dz_1 \wedge dz_2} \oplus \underbrace{\Lambda^{1,1}}_{\substack{dz_1 \wedge d\bar{z}_1 \\ dz_1 \wedge d\bar{z}_2 \\ dz_2 \wedge d\bar{z}_1 \\ dz_2 \wedge d\bar{z}_2}} \oplus \underbrace{\Lambda^{0,2}}_{d\bar{z}_1 \wedge d\bar{z}_2}$$

1   4   1

4

$$\boxed{\Lambda^3(T_p^*M \otimes \mathbb{C})} = \underbrace{\Lambda^{3,0}}_{0} \oplus \underbrace{\Lambda^{2,1}}_{\substack{dz_1 \wedge dz_2 \wedge d\bar{z}_1 \\ dz_1 \wedge dz_2 \wedge d\bar{z}_2}} \oplus \underbrace{\Lambda^{1,2}}_{\substack{dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \\ dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2}} \oplus \underbrace{\Lambda^{0,3}}_{0}$$

2   2

$$\Lambda^4(T_p^*M \otimes \mathbb{C}) = \underbrace{\Lambda^{4,0}}_{0} \oplus \underbrace{\Lambda^{3,1}}_{0} \oplus \underbrace{\Lambda^{2,2}}_{dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2} \oplus \underbrace{\Lambda^{1,3}}_{0} \oplus \underbrace{\Lambda^{0,4}}_{0}.$$

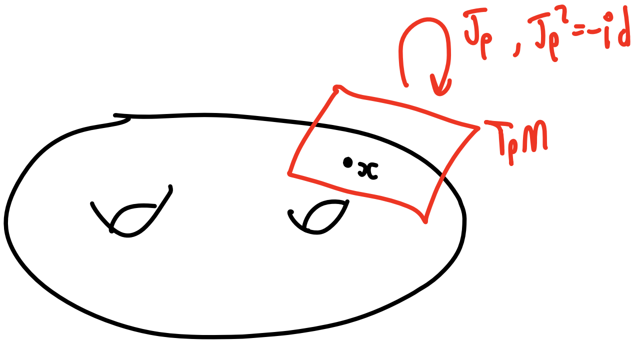
eg. a 3-form on M lodes locally like

$$\omega = f_1\, dz_1 \wedge dz_2 \wedge d\bar{z}_1 + f_2\, dz_1 \wedge dz_2 \wedge d\bar{z}_2$$
$$+ f_3\, dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 + f_4\, dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2$$

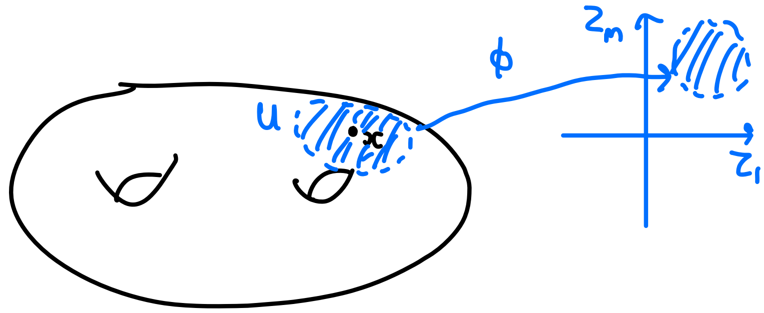
$$f_1(x_1, x_2, y_1, y_2) = f(z_1, z_2, \bar{z}_1, \bar{z}_2)$$

2.5. Almost-complex vs Complex manifolds

When can we upgrade an almost complex-manifold (M,J) to a complex manifold?



almost-complex manifold (M,J)
(infinitesimal structure ... lives on tangent spaces)



complex manifold = holomorphic atlas
(local structure ... lives on open sets)

Related question:

We've seen that the complex-valued k-forms on every almost-complex manifold (M,J) decompose as

$$\Omega^k(M,\mathbb{C}) = \bigoplus_{a+b=k} \Omega^{a,b}(M)$$

Does the exterior derivative map

$$d : \Omega^k(M,\mathbb{C}) \longrightarrow \Omega^{k+1}(M,\mathbb{C})$$

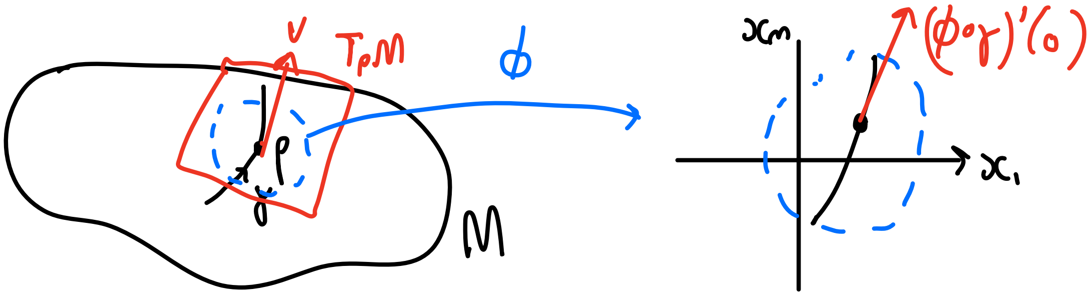
respect this decomposition?

To answer this, we need to know some material about vector fields I skipped earlier.

Recall that for us, a vector field is a smooth selection

$$X_p \in T_pM$$

of a vector from the tangent space at each $p \in M$. And that $v \in T_pM$ is defined as an <u>equivalence class of curves</u> going through p:



In particular, every vector field X gives us a linear operator

I'm also calling it X for convenience

$$X : C^\infty(M) \longrightarrow C^\infty(M)$$

defined by

$$X(f)(p) := \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) \qquad [\gamma] = X_p.$$

These linear maps satisfy the <u>Leibniz rule</u>:

$$X(fg) = X(f)g + fX(g).$$

This gives us an alternative, operator theoretic, way to <u>define</u> vector fields!

Lemma A smooth vector field on M is the same thing as a linear operator

$$X : C^\infty(M) \longrightarrow C^\infty(M)$$

which satisfies the Leibniz rule.

Exercise 1 Fill in the details of the proof! C.f. Looijenga Prop 2.1

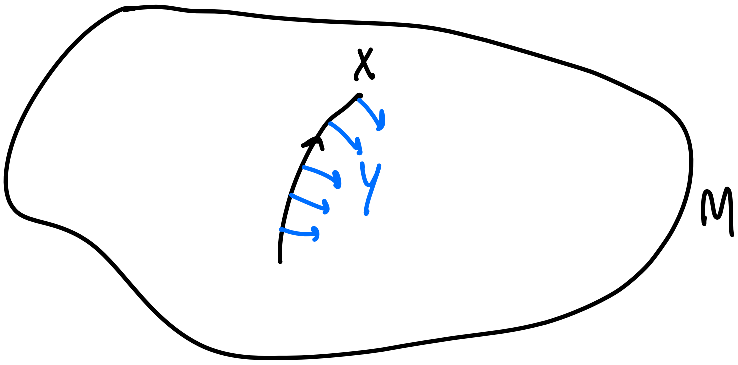
This operator-theoretic viewpoint is very useful for Lie brackets:

Definition The <u>Lie bracket</u> of two smooth vector fields X and Y on a smooth manifold M is

$$[X,Y](f) := XY(f) - YX(f)$$

Exercise 2 Show that $[X,Y]$ is a vector field! Hint: use the operator theoretic description

The Lie bracket $[X,Y]_p$ measures the change in Y as we move along the integral curves of X (but we don't need this right now!)



The Lie bracket of vector fields also gives us a new, coordinate free way to define the exterior derivative of differential forms! Recall that currently our formula is:

$$d := \sum_{j=1}^{m} \frac{\partial}{\partial x_j}(\cdots)\, dx_j \wedge \text{" "}$$

That is, if $\omega \in \Omega^k(M)$, to compute $d\omega$ we choose a coordinate chart $(x_1, \cdots, x_m)$ locally, so that

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le m} \omega_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and then we set

$$d\omega = \sum_{1 \le i_1 < \cdots < i_k \le m} \sum_{j=1}^{m} \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Coordinate-free formula for exterior derivative $d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

$$d\omega(X_0, \cdots X_k) = \sum_i (-1)^i X_i\left(\omega(X_0, \cdots, \hat{X}_i, \cdots, X_k)\right) + \sum_{i<j} \omega([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k)$$

Exercise 3 Check that this formula at least agrees with our previous formula, in a coordinate chart.

We are now ready to give the answer to our first question about when an almost-complex manifold can be upgraded to a complex manifold!

Newlander-Nirenberg Theorem An almost-complex manifold (M, J) can be upgraded to a complex manifold (i.e. equipped with a compatible holomorphic atlas) if and only if J is integrable in the sense that:

$$\forall \quad X, Y \in C^\infty(M, T^{1,0}M), \qquad [X, Y] \in C^\infty(M, T^{1,0}M)$$

We can also give the answer to our second question about how the exterior derivative interacts with the decomposition of $\Omega^k(M)$:

Proposition Let (M, J) be an almost-complex manifold. The following are equivalent:

1. J is integrable

2. $d\left(\Omega^{1,0}(M)\right) \subseteq \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$

3. $d\left(\Omega^{p,q}(M)\right) \subseteq \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ for all p, q.

Proof (1) $\Leftrightarrow$ (2) Let $\omega \in \Omega^{1,0}(M)$, $X, Y \in C^\infty(M, T^{0,1}M)$

Then

$$d\omega(X,Y) = X(\underbrace{\omega(Y)}_{=0}) - Y(\underbrace{\omega(X)}_{=0}) - \omega([X,Y])$$

$$= -\omega([X,Y])$$

$$\therefore \quad d\omega(X,Y) = 0 \iff [X,Y]^{1,0} \in \ker \omega$$

$$\therefore \quad d\Omega^{1,0} \subseteq \Omega^{2,0} \oplus \Omega^{1,1} \cancel{\oplus \Omega^{0,2}}$$

$$\iff \text{for all } X, Y \in C^\infty(M, T^{0,1}M),$$

$$[X,Y]^{1,0} = 0$$

i.e. $[X,Y] \in C^\infty(M, T^{0,1}M)$

(2) $\Leftarrow$ (3) Follows from Leibniz formula for d on k-forms

$\boxed{\text{Exercise 4. Check this!}}$

$$\omega \in \Omega^k(M), \quad \eta \in \Omega^\ell(M)$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

$\square$

$\square$

As a corollary, we see that every 2-dimensional almost-complex manifold (M, J) is integrable ... since we must have

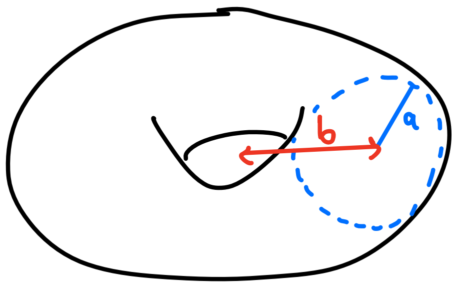
$$d\,\Omega^{1,0}(M) \subseteq \underbrace{\Omega^{2,0}(M)}_{=0} \oplus \Omega^{1,1}(M) \oplus \underbrace{\Omega^{0,2}(M)}_{=0}$$

as $\Omega^{2,0}(M) = \Omega^{0,2}(M) = 0$!

In particular, every surface embedded in $\mathbb{R}^3$ has a canonical complex structure! See example from Section 2.2.

<u>Puzzle</u> Describe geometrically the holomorphic coordinates on such a surface (we only know how to do this for a <u>minimal</u> surface...)

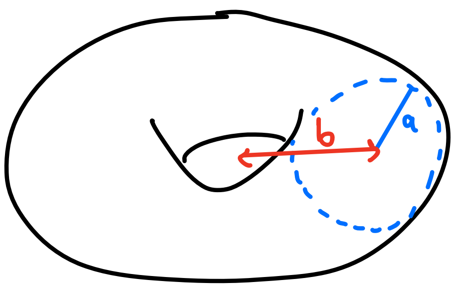
Puzzle Consider the "round torus" $T_{a,b} \subseteq \mathbb{R}^3$, as a complex manifold:



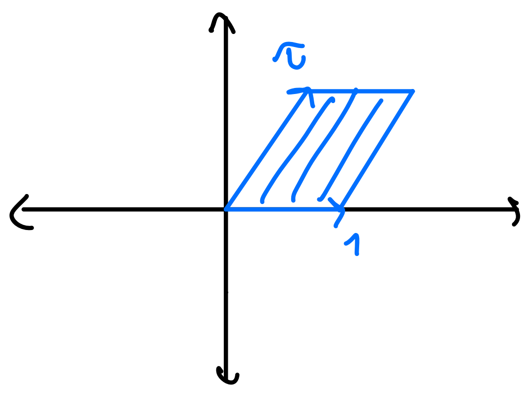
We know it is holomorphically diffeomorphic to

$$\mathbb{C}/\mathbb{Z}\langle 1, \tau \rangle$$

for some $\tau \in \mathbb{H}$. Determine $\tau(a,b)$.



$\cong$



2.6. The Dolbeault operators

So, on a complex manifold, we have local holomorphic coordinates $z_1, \cdots, z_m$ and a (p,q)-form looks like

$$\omega = \sum_{\substack{1 \leq i_1 < \cdots < i_p \leq m \\ 1 \leq j_1 < \cdots < j_q \leq m}} \omega_{i_1, \cdots, i_p, j_1, \cdots, j_q}(z_1, \cdots, z_m, \bar{z}_1, \cdots, \bar{z}_m)\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

We can calculate $d\omega$ in these holomorphic coordinates,

$$d = \underbrace{\sum_{i=1}^{m} \frac{\partial}{\partial z_i}(\cdots)\, dz_i \wedge (\cdots)}_{\partial} + \underbrace{\sum_{j=1}^{m} \frac{\partial}{\partial \bar{z}_i}(\cdots)\, d\bar{z}_i \wedge (\cdots)}_{\bar{\partial}}$$

In other words, since

$$d : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$$

we can write $d = \underset{\text{"del"}}{\partial} + \bar{\partial}$ where:

the Dolbeault operators

$$\partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M)$$
$$\bar{\partial} : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M)$$

Lemma Let $f: M \longrightarrow \mathbb{C}$ be a smooth function. The following are equivalent:

$\in C^\infty(M, T^{0,1}M)$

1. f is holomorphic
2. $\bar\partial f = 0$
3. for every $Z \in C^\infty(M, T^{1,0}M)$, $\boxed{\bar Z}(f) = 0$.

Proof $(1) \Leftrightarrow (2)$. $\bar\partial f = \sum_{i=1}^{m} \frac{\partial f}{\partial \bar z_i} d\bar z_i$

$\therefore \quad \bar\partial f = 0 \Leftrightarrow \frac{\partial f}{\partial \bar z_i} = 0 \quad \forall\, i = 1 \dots m$

$\Leftrightarrow f$ is holomorphic

$(2) \Leftrightarrow (3)$: $\bar Z(f) = df(\bar Z)$

$= \bar\partial f(\bar Z)$

$\therefore \quad \bar\partial f = 0 \Leftrightarrow \forall\, Z \in C^\infty(M, T^{1,0}M), \ \bar Z(f) = 0.$

$\square$

Lemma $\partial^2 = 0$, $\partial\bar\partial + \bar\partial\partial = 0$, $\bar\partial^2 = 0$. (recall: $d^2 = 0$)

Proof $d = \partial + \bar\partial$

$\therefore d^2 = 0 \Leftrightarrow (\partial + \bar\partial)^2 = 0$

i.e. $\underbrace{\partial^2}_{=0} + \underbrace{\partial\bar\partial + \bar\partial\partial}_{=0} + \underbrace{\bar\partial^2}_{=0} = 0$

$\therefore$

$$\begin{tikzcd} & & \Omega^{p+2,q} \\ & \Omega^{p+1,q} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"'] & \\ \Omega^{p,q} \arrow[ur, "\partial"] & & \Omega^{p+1,q+1} \end{tikzcd}$$

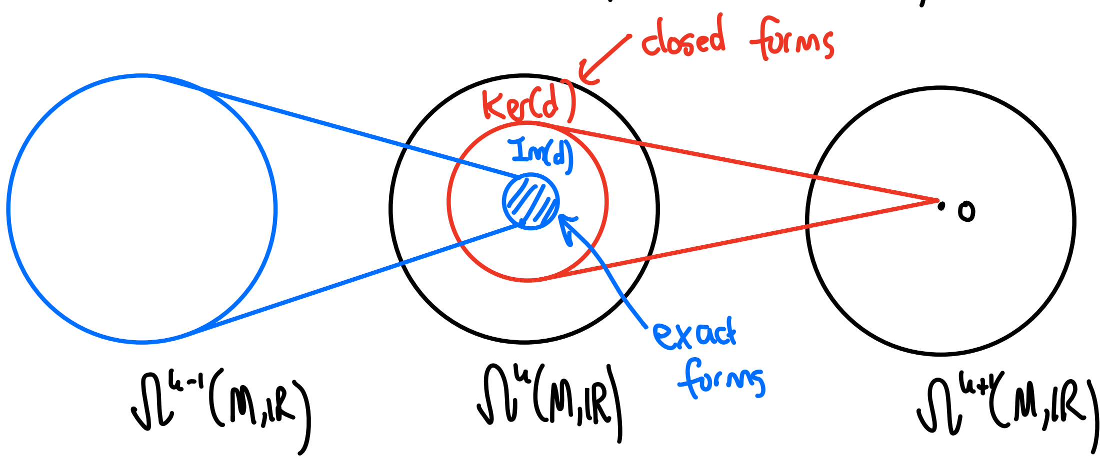
$$\bar\partial \searrow \Omega^{p,q+1}$$

$$\bar\partial \searrow \Omega^{p,q+2}$$

$\square$

Recall: on a smooth manifold, the exterior derivative

$$\cdots \longrightarrow \Omega^{k-1}(M,\mathbb{R}) \xrightarrow{d} \Omega^{k}(M,\mathbb{R}) \xrightarrow{d} \Omega^{k+1}(M,\mathbb{R}) \longrightarrow \cdots$$



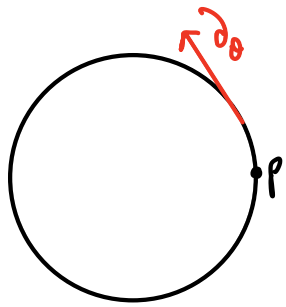
$\therefore$ $\mathrm{Im}(d) \subseteq \mathrm{Ker}(d)$

satisfies $d^2 = 0$, and we define the <u>De Rham cohomology</u> as

$$H^k_{DR}(M) := \frac{\mathrm{Ker}\left(d : \Omega^k(M,\mathbb{R}) \longrightarrow \Omega^{k+1}(M,\mathbb{R})\right)}{\mathrm{Im}\left(d : \Omega^{k-1}(M,\mathbb{R}) \longrightarrow \Omega^{k}(M,\mathbb{R})\right)}$$

> <u>Poincaré lemma</u> Every d-closed k-form ω on a smooth manifold is <u>locally exact</u>, i.e. locally on a small enough open set U,
>
> $$\omega = d\alpha \quad \text{on } U.$$

But this is not the case globally on M, eg.



$M = S^1$

$d\omega = 0$ ... why ?

$\omega \in \Omega^1(S^1)$, $\omega(\partial_\theta) = 1$

$\omega \neq df$ since otherwise

$$\int_{S^1} \omega = \int_p^p df = \underset{=0}{f(p) - f(p)}$$

but $\displaystyle\int_{S^1} \omega = 2\pi.$

Indeed, $H^1(S^1) = \mathbb{R}[\omega]$.

Similarly:

We define the <u>Dolbeault cohomology</u> groups of a complex manifold M as

$$H^{p,q}(M) = \frac{\mathrm{Ker}\left(\partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M)\right)}{\mathrm{Im}\left(\partial : \Omega^{p-1,q}(M) \longrightarrow \Omega^{p,q}(M)\right)}$$

The $\bar{\partial}$-Poincaré lemma A $\bar{\partial}$-closed form on a complex manifold is locally $\bar{\partial}$-exact.

$\omega \in \Omega^{1,0}(M) \quad \bar{\partial}\omega = 0.$

Puzzle Describe ^explicitly the holomorphic 1-forms on the round torus $T_{a,b}$.

To a complex-manifold aficicionado,

$$\operatorname{genus}(\underset{\substack{\uparrow \\ \text{Riemann} \\ \text{surface}}}{M}) := \dim \{\text{holomorphic 1-forms on } M\}$$
$$= \dim \operatorname{Ker}\left(\bar{\partial} : \Omega^{1,0} \longrightarrow \Omega^{1,1}\right)$$

$$\underbrace{C^\infty(U, \mathbb{R})}_{\subseteq \Omega^{0,0}} \xrightarrow{\bar{\partial}} \Omega^{0,1} \xrightarrow{\partial} \underbrace{\Omega^{1,1}}_{\alpha}$$

Lemma ($i\partial\bar\partial$ lemma) Let $\alpha \in \Omega^2(M,\mathbb{R})$. Then

$$d\alpha = 0 \quad \text{and} \quad \alpha \in \Omega^{1,1}(M) \iff \text{Locally, } \alpha = i\partial\bar\partial\phi \text{ for some } \phi \in C^\infty(U,\mathbb{R}).$$

i.e. $\alpha \in \Omega^2(M,\mathbb{R})$

$\subseteq \Omega^2(M,\mathbb{C}) = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$

Proof ($\Leftarrow$) Suppose $\alpha = i\partial\bar\partial\phi$ locally, for some

$$\phi \in C^\infty(U,\mathbb{R}).$$

$$\begin{aligned} d\alpha &= (\partial + \bar\partial)\alpha \\ &= i(\partial+\bar\partial)\partial\bar\partial\phi \\ &= i\Big(\underbrace{\partial\partial}_{=0}\bar\partial\phi + \underbrace{\bar\partial\partial}_{=-\partial\bar\partial}\bar\partial\phi\Big) \\ &= 0 \end{aligned}$$

$\boxed{\bar\partial\bar\partial} = 0$

bar: $\Omega^{p,q} \longrightarrow \Omega^{q,p}$

Also,

$$\begin{aligned} \bar\alpha &= -i\bar\partial\partial\phi \\ &= i\partial\bar\partial\phi \\ &= \alpha \end{aligned}$$

$\therefore$ α is real-valued.

Also, clearly since $\alpha = i\partial\bar\partial\phi$, we have $\alpha \in \Omega^{1,1}(M)$.

$(\Longrightarrow)$ Suppose $\alpha \in \Omega^2(M, \mathbb{R})$ is closed (i.e. $d\alpha = 0$)

$\therefore$ From usual Poincare lemma, we know that locally, there exists 1-form β on U s.t.

$$\alpha = d\beta \qquad \text{on } U.$$

We can decompose

$$\beta \in \Omega^1(M, \mathbb{R}) \subseteq \Omega^1(M, \mathbb{C}) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$$

$$\beta = \beta^{1,0} + \beta^{0,1}$$

$$\therefore \quad \underbrace{\alpha}_{\in \Omega^{1,1}} = \underbrace{\partial \beta^{1,0}}_{\substack{\in \Omega^{2,0} \\ \therefore = 0}} + \underbrace{\bar\partial \beta^{1,0} + \partial \beta^{0,1}}_{\in \Omega^{1,1}} + \underbrace{\bar\partial \beta^{0,1}}_{\substack{\in \Omega^{0,2} \\ \therefore = 0}}$$

Since $\bar\partial \beta^{0,1} = 0$, we know from the $\bar\partial$-Poincare lemma that locally we can write

$$\beta^{0,1} = \bar\partial f \qquad f \in C^\infty(U, \mathbb{C})$$

But $\beta = \bar\beta \Rightarrow \beta^{1,0} = \overline{\beta^{0,1}}$

$$= \overline{\bar\partial f} = \partial \bar f$$

$\therefore$

$$\therefore \quad \alpha = \bar\partial\partial f + \partial\bar\partial \bar f$$
$$= \partial\bar\partial(f - \bar f)$$
$$= i\partial\bar\partial\phi \quad \text{where} \quad \phi = 2\,\mathrm{Im} f$$

$\square$

<u>Another look</u> Suppose I have

$$\alpha = i\partial\bar\partial\phi \in \Omega^{1,1}(U), \qquad \phi \in C^\infty(U,\mathbb{R}).$$

So, if $z_i = x_i + iy_i$ are local holomorphic coordinates on U, then

$$\alpha = i\partial \sum_{k=1}^{m} \frac{\partial\phi}{\partial\bar z_k}\, d\bar z_k$$
$$= i\sum_{j,k} \underbrace{\frac{\partial^2\phi}{\partial z_j\partial\bar z_k}}\, \underbrace{dz_j}_{= dx_j + i\,dy_j} \wedge \underbrace{d\bar z_k}_{= dx_k - i\,dy_k}$$

$\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$

$\partial_{\bar z_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$

$$\therefore \quad \frac{\partial^2\phi}{\partial z_j\partial\bar z_k} = \frac{1}{4}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)\left(\frac{\partial\phi}{\partial x_k} + i\frac{\partial\phi}{\partial y_k}\right)$$

$$= \tfrac{1}{4}\left[\frac{\partial^2\phi}{\partial x_j \partial x_k} + \frac{\partial^2\phi}{\partial y_j \partial y_k} + i\left(\frac{\partial^2\phi}{\partial x_j \partial y_k} - \frac{\partial^2\phi}{\partial y_j \partial x_k}\right)\right]$$

$$= \phi_{jk} + i\psi_{jk}$$

Note: $\phi_{jk} = \phi_{kj}$, $\psi_{jk} = -\psi_{kj}$

$$dz_j \wedge d\bar{z}_k = (dx_j + i\,dy_j) \wedge (dx_k - i\,dy_k)$$

$$= dx_j \wedge dx_k + dy_j \wedge dy_k + i(dy_j \wedge dx_k - dx_j \wedge dy_k)$$

$$\therefore \quad \alpha = i\partial\bar{\partial}\phi \qquad \overline{\alpha} = -i\sum_{j,k}\phi_{jk}$$

$$= i\sum_{j,k}\left(\phi_{jk} + i\psi_{jk}\right)\left(dx_j \wedge dx_k + dy_j \wedge dy_k + i(dy_j \wedge dx_k - dx_j \wedge dy_k)\right)$$

$$= i\sum_{j,k}\Big[\overbrace{\phi_{jk}(dx_j \wedge dx_k + dy_j \wedge dy_k)}^{=0 \quad j\leftrightarrow k} - \overbrace{\psi_{jk}(dy_j \wedge dx_k - dx_j \wedge dy_k)}^{=0 \quad j\leftrightarrow k}$$

$$+ i\left[\psi_{jk}(dx_j \wedge dx_k + dy_j \wedge dy_k) + \phi_{jk}(dy_j \wedge dx_k - dx_j \wedge dy_k)\right]\Big]$$

$$= -\sum_{j,k}\left[\psi_{jk}(dx_j \wedge dx_k + dy_j \wedge dy_k) + \phi_{jk}(dy_j \wedge dx_k - dx_j \wedge dy_k)\right]$$

which is clearly a real 2-form, and which is also closed (check), confirming the theorem.

In 2 real dimensions: If $\phi \in C^\infty(U, \mathbb{R})$, then

$$
\begin{aligned}
i\partial\bar{\partial}\phi &= -\phi_{1\bar{1}}\left(dy \wedge dx - dx \wedge dy\right) \\
&= 2\left(\frac{1}{4}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right) dx \wedge dy \\
&= \tfrac{1}{2}\underbrace{\Delta\phi}_{\text{Laplacian of } \phi.}\, dx \wedge dy.
\end{aligned}
$$

So the theorem is saying that any closed real 2-form of type (1,1), ie.

$$\alpha = i\underbrace{f(x,y)}_{\text{arbitrary smooth real function}}\, dz \wedge d\bar{z}$$

locally, can be expressed as

$$\alpha = \frac{1}{2}\left(\phi_{xx} + \phi_{yy}\right) dx \wedge dy$$

In other words

$$
\begin{aligned}
\alpha &= f(x,y)\, i\, dz \wedge d\bar{z} \\
&= f(x,y)\, 2\, dx \wedge dy \\
&= \frac{1}{2}\left(\phi_{xx} + \phi_{yy}\right) dx \wedge dy
\end{aligned}
$$

So the theorem is saying that any real smooth function

$$f(x,y)$$

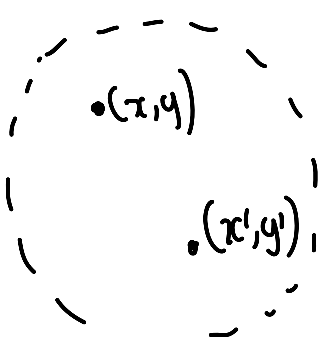
can be expressed as the Laplacian of some ϕ locally,

$$f = \frac{1}{4}\Delta\phi$$

which we know is actually true, i.e. we need to solve the PDE (<u>Poisson's equation</u>)

$$\Delta\phi = 4f \qquad f \in C^\infty(U,\mathbb{R}) \qquad (*)$$

for ϕ. We can indeed solve Poisson's equation! For instance, suppose that $U = D$, the open unit disk, and suppose f extends continuously to $\partial\bar{D} = S^1$. Then the solution to $(*)$ is obtained using <u>Green's functions</u> and the <u>Poisson kernel</u>:



$$\phi(x,y) = \frac{1}{\pi}\int_{(x',y')\in D} \underbrace{G(x,y,x',y')}_{\text{Green's function on unit disk}} f(x',y')\,dx'dy'$$

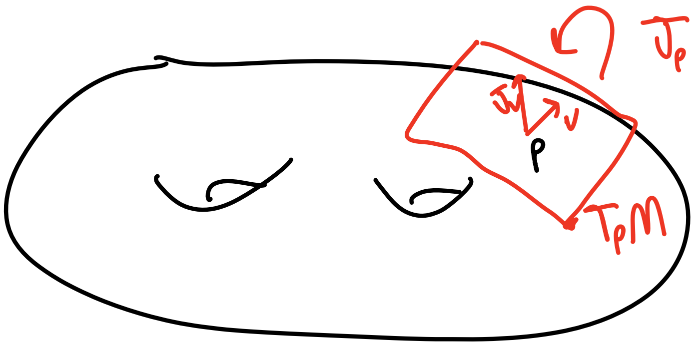
$$+ \frac{2}{\pi}\int_{x'\in\partial\bar{D}} \underbrace{P(x,x')}_{\text{Poisson kernel on unit disk}} f(x')\,dx'$$

2.7. Kähler manifolds

(M, J) complex manifold.

So we have at each (real) tangent space T_pM,

$$J_p : T_pM \longrightarrow T_pM, \qquad J_p^2 = -\mathrm{id}$$



$x_1, \dots, x_n$ local coordinates

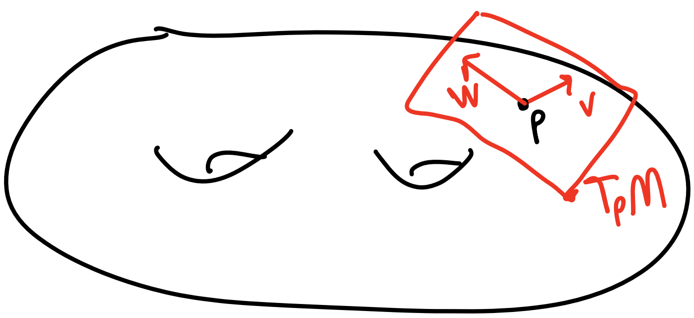
$g_{ij} = g_{ji}$

$$\boxed{g = \sum_{i,j} g_{ij}\, dx_i \otimes dx_j}$$

Recall that a <u>Riemannian metric</u> g on a smooth manifold M is a (smooth) selection of an <u>inner product</u> g_p on each tangent space T_pM, i.e. $g \in C^\infty(M, T^*M \otimes T^*M)$, such that for all $p \in M$,

$$g_p : T_pM \otimes T_pM \longrightarrow \mathbb{R}$$

is <u>symmetric</u> and <u>positive-definite</u> $\big(g_p(v,v) \geq 0$ and $g_p(v,v) = 0 \Leftrightarrow v = 0 \big)$



$g_p(v,w)$ inner product

Lemma Let

$$(\cdot\,,\,\cdot) : V \otimes V \longrightarrow \mathbb{R}$$

be a symmetric bilinear map on a real vector space, satisfying

$$(v,v) \geq 0 \quad \text{for all } v \in V$$

Then the following are equivalent:

$$\begin{aligned} V &\longrightarrow V^\vee \\ v &\longmapsto (v,-) \end{aligned} \qquad \text{injective}$$

① $(\cdot\,,\,\cdot)$ is nondegenerate, i.e. $(v,w)=0 \;\forall\, w \Rightarrow v=0$.

② $(\cdot\,,\,\cdot)$ is positive-definite, i.e. $(v,v)=0 \Leftrightarrow v=0$

Proof $\boxed{\text{Exercise 1}}$

Given (M, J) and a Riemannian metric g, we say that g is <u>compatible with J</u> if J preserves the inner product, i.e.

$$g(JX, JY) = g(X, Y)$$

on each tangent space.

Given a compatible Riemannian metric g on (M, J), we define its <u>fundamental form</u> as

$$\omega(X, Y) = g(JX, Y) \qquad X, Y \in TM.$$

Note also how ω gets along with J:

$$\begin{aligned}
\omega(JX, JY) &= g(J^2X, JY) \\
&= -g(X, JY) \\
&= -g(JX, J^2Y) \\
&= g(JX, Y) \\
&= \omega(X, Y).
\end{aligned}$$

<u>Lemma</u> ω is a real $(1,1)$-form.

<u>Proof</u> Firstly, ω is actually a 2-form (i.e. antisymmetric), since

$$\begin{aligned}\omega(Y,X) &= g(JY, X) \\ &= g(JJY, JX) \qquad [g \text{ compatible with } J] \\ &= g(-Y, JX) \\ &= -g(JX, Y) \\ &= -\omega(X,Y).\end{aligned}$$

In other words, J preserves the fundamental form.

Is ω of type $(1,1)$? We need to check that

① $\omega(X,Y) = 0$ if $X \in T^{1,0}M$, $Y \in T^{1,0}M$

and ② $\omega(X,Y) = 0$ if $X \in T^{0,1}M$, $Y \in T^{0,1}M$

① : Well, we know that for all $X, Y \in T_pM$,

$$\omega(JX, JY) = \omega(X,Y).$$

But, if $X, Y \in T^{1,0}M$, then $JX = iX$, $JY = iY$

$$\therefore \omega(JX, JY) = \omega(iX, iY) = -\omega(X,Y).$$

$$\therefore \omega(X,Y) = -\omega(X,Y) \Rightarrow \omega(X,Y) = 0$$

(2) is similar.

$\square$

Similarly, g is a $(1,1)$-tensor, in the sense that

(1) $g(X,Y) = 0$ if $X, Y \in T_p^{1,0}M$

(2) $g(X,Y) = 0$ if $X, Y \in T_p^{0,1}M$.

<u>Proof</u> We always have $g(JX, JY) = g(X,Y)$.

So if $JX = iX$, $JY = iY$ then $LHS = -RHS \Rightarrow LHS = RHS = 0$ $\square$

Example local calculation Let M be a complex manifold with local coordinates $z_1, \dots, z_m$.

Since $\omega \in \Omega^{1,1}(M)$, we have locally that

$$\omega = i \sum_{j<k} \underbrace{\omega_{jk}}_{\in \mathbb{C}} dz_j \wedge d\bar{z}_k$$

$dz \wedge d\bar{z}$
$= -2i\, dx \wedge dy$

where
$$\begin{aligned} \omega_{jk} &= -i\,\omega(\partial_{z_j}, \partial_{\bar{z}_k}) \\ &= -i\,g(J\partial_{z_j}, \partial_{\bar{z}_k}) \\ &= g(\partial_{z_j}, \partial_{\bar{z}_k}) =: \underbrace{g_{jk}}_{\in \mathbb{C}}, \end{aligned}$$

Note: $g_{kj} = \overline{g_{jk}}$, so that (g_{jk}) - or equivalently (ω_{jk}) - is a positive definite $m \times m$ Hermitian matrix.

So: (M, J) almost complex manifold

g Riemannian metric on M, compatible with J

$\rightsquigarrow \omega \in \Omega^2(M)$ ... fundamental form.

Recall that a <u>symplectic form</u> on a smooth manifold M^{2m} is a 2-form $\omega \in \Omega^2(M)$ satisfying:

① ω is <u>nondegenerate</u> at each $p \in M$:

$$\omega_p(X, Y) = 0 \quad \forall\, Y \in T_pM \iff X = 0.$$

② ω is <u>closed</u>, i.e. $d\omega = 0$

The fundamental form ω of a compatible Riemannian metric g is nondegenerate, since g is nondegenerate. Is it closed?

<u>Definition</u> A compatible Riemannian metric g on an a complex manifold (M, J) is called a <u>Kähler metric</u> if its fundamental form ω is closed.

A <u>Kähler manifold</u> is a complex manifold equipped with a Kähler metric.

<u>Exercise 2</u> Let V be a real finite-dimensional vector space and $\omega \in \Lambda^2 V^*$. Prove that the following are equivalent:

1. ω is <u>nondegenerate</u> i.e.

$$\omega(v,w) = 0 \ \forall w \iff v = 0$$

2. $$\underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{m} \neq 0$$

$\in \Lambda^{2m} V^* \cong \mathbb{R}$

$\left(m = \frac{1}{2} \dim V \right)$

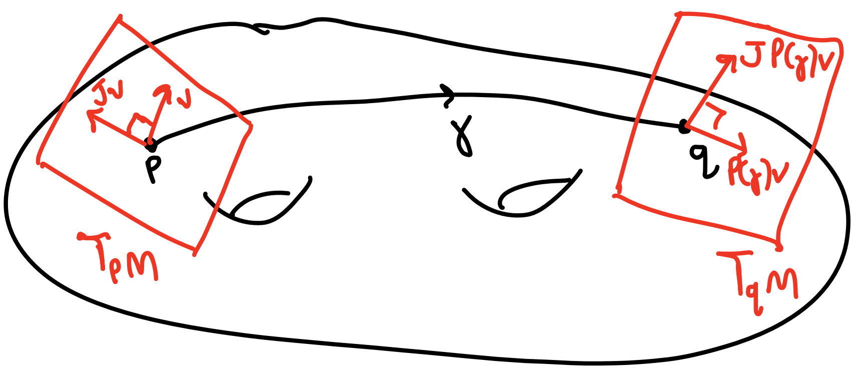
$\dim V = 2m$

We can phrase this condition on the Riemannian metric g entirely in terms of (M, J)

Theorem Let (M, J) be a complex manifold and g a Riemannian metric on M compatible with J. Let ω be the fundamental form of g. The following are equivalent:

1. $d\omega = 0$
2. $\nabla J = 0$, where ∇ is the Levi-Civita connection on M associated to g.

in other words, parallel transport of g is compatible with J



$$\begin{tikzcd} T_pM \arrow[r, "P(\gamma)"] \arrow[d, "J_p"'] & T_qM \arrow[d, "J_q"] \\ T_pM \arrow[r, "P(\gamma)"] & T_q(M) \end{tikzcd}$$

So, if (M, J, g) is a Kähler manifold, then, since the fundamental form $\omega(X,Y) := g(JX, Y)$ is a $(1,1)$-form, we can write locally

$$\omega = i \partial \bar{\partial} \phi \quad , \quad \phi \in C^\infty(U, \mathbb{R})$$

This means we can express the metric as:

$g \rightsquigarrow \omega(X,Y) := g(JX,Y)$

NB: the metric is a $(1,1)$-tensor! $\longrightarrow$

$$g = \sum_{j,k} g_{j k} \, dz_j \otimes d\bar{z}_k$$

$\omega \rightsquigarrow g(X,Y) = \omega(X, JY)$

where

$$\begin{aligned} g_{jk} &= g(\partial_{z_j}, \partial_{\bar{z}_k}) \\ &= \omega(\partial_{z_j}, J \partial_{\bar{z}_k}) \\ &= -i \, \omega(\partial_{z_j}, \partial_{\bar{z}_k}) \\ &= (-i)(i \partial \bar{\partial} \phi)(\partial_{z_j}, \partial_{\bar{z}_k}) \\ &= \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \end{aligned}$$

So a Kähler metric is constructed locally from a <u>single smooth function</u>. (Usually, a Riemannian metric is constructed locally from several independent functions).

At the moment we have:

- (M, J) complex manifold

$\rightarrow$ g Riemannian metric is <u>Kähler</u> if
  - $g(JX, JY) = g(X, Y)$
  - $\omega(X, Y) := g(JX, Y)$ is closed.

The other point of view is to start with:

- (M, ω) a <u>symplectic manifold</u>

$\rightarrow$ J integrable almost complex structure s.t.
  - $\omega(JX, JY) = \omega(X, Y)$

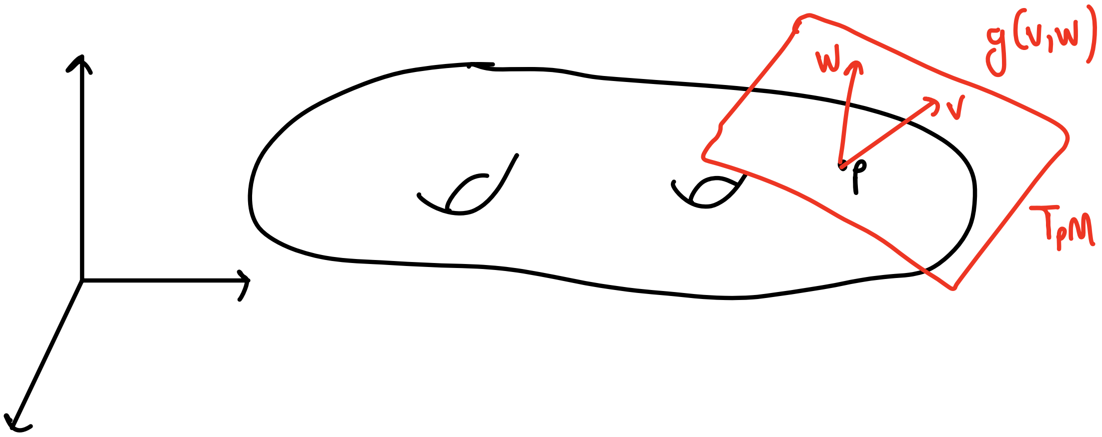
Then $g(X, Y) := \omega(X, JY)$ is a Kähler metric.

<u>Exercise 3.</u> Prove this

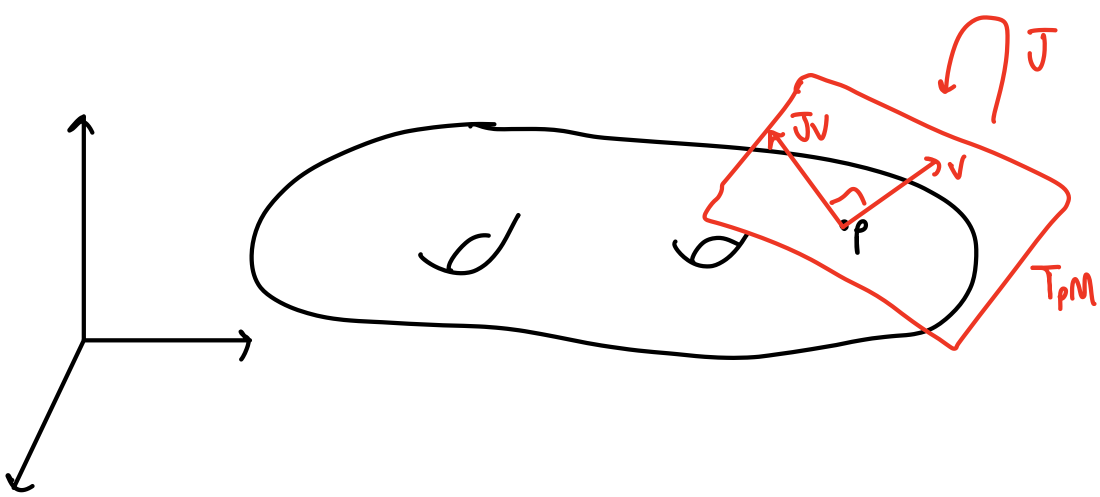
So, said differently, a Kähler manifold is (M, J, g, ω) where:

- J is an integrable almost complex structure
- g is a Riemannian metric compatible with J and ω
- ω is a symplectic form compatible with J and g.

<u>Example 1</u> Every embedded oriented surface $M \subseteq \mathbb{R}^3$ naturally inherits a Riemannian metric from the ambient space $\mathbb{R}^3$:

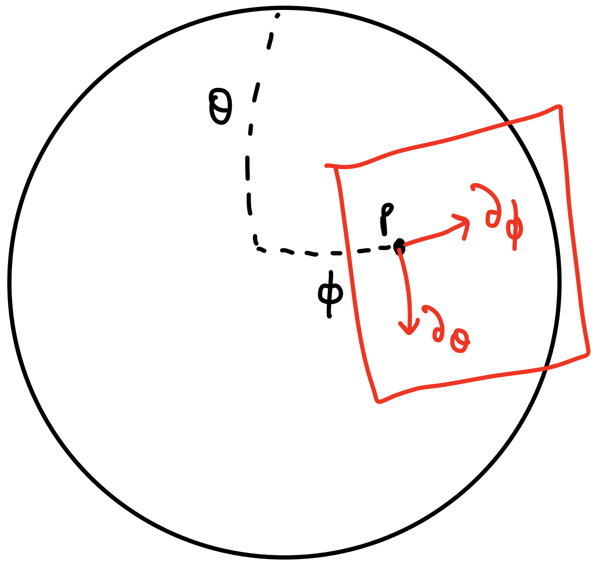


And we know that M similarly inherits a natural integrable complex structure J.



Is g a Kähler metric for (M, J)? Yes, because its fundamental form $\omega \in \Omega^2(M)$ must be closed, i.e. $d\omega = 0$, since there are no non-zero 3-forms on M!

<u>Example 2</u> Let's work this out explicitly for S^2, in the (θ, ϕ) coordinate system:



$$p(\theta, \phi) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\therefore \quad \partial_\theta = \frac{\partial p}{\partial \theta} = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)$$

$$\partial_\phi = \frac{\partial p}{\partial \phi} = (-\sin\theta\sin\phi,\ \sin\theta\cos\phi,\ 0)$$

$$g_{\theta\theta} = g(\partial_\theta, \partial_\theta) = 1$$

$$g_{\theta\phi} = g(\partial_\theta, \partial_\phi) = 0 \qquad g \to \begin{array}{c|cc} & \theta & \phi \\ \hline \theta & 1 & 0 \\ \phi & 0 & \sin^2\theta \end{array}$$

$$g_{\phi\phi} = g(\partial_\phi, \partial_\phi) = \sin^2\theta \qquad \therefore\ vol_g := e^\theta \wedge e^\phi = \sin\theta\, d\theta \wedge d\phi$$

$$e^\theta = d\theta \qquad e^\phi = \sin\theta\, d\phi$$

Set $e_\theta = \dfrac{\partial_\theta}{\sqrt{(\partial_\theta, \partial_\theta)}} = \partial_\theta$, $\quad e_\phi = \dfrac{\partial_\phi}{\sqrt{(\partial_\phi, \partial_\phi)}} = \dfrac{\partial_\phi}{\sin\theta}$

And so, since $e_\theta \overset{J}{\longmapsto} e_\phi$, $e_\phi \overset{J}{\longmapsto} -e_\theta$,

we have:
$$J = \begin{bmatrix} 0 & -\sin\theta \\ \frac{1}{\sin\theta} & 0 \end{bmatrix}$$

$$J(\partial_\theta) = \frac{\partial_\phi}{\sin\theta}, \qquad J(\partial_\phi) = -\sin\theta\, \partial_\theta$$

The fundamental form of g is

$$\omega = \omega_{\theta\phi}\, d\theta \wedge d\phi$$

where $\omega_{\theta\phi} = g(J\partial_\theta, \partial_\phi) = g\left(\dfrac{\partial_\phi}{\sin\theta}, \partial_\phi\right)$

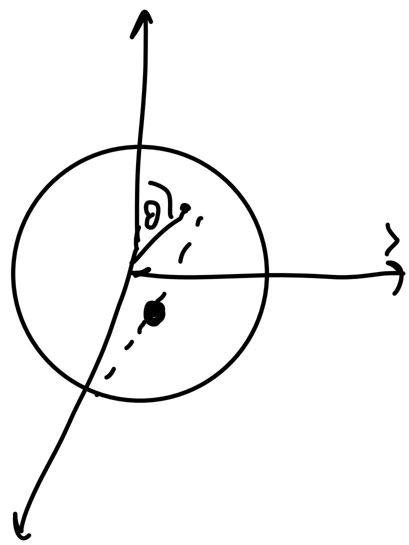
$$= \frac{1}{\sin\theta} \cdot \sin^2\theta \qquad = \sin\theta$$

$$\therefore \quad \omega = \sin\theta \; d\theta \wedge d\phi$$

By definition, this is the standard area 2-form of S^2. Note:

$$\int_{S^2} \omega = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \cdot \int_{\theta=0}^{\pi} \sin\theta$$



$$= 4\pi$$

which is the area of S^2.

Let's write ω in complex coordinates:

$$z = \tan\theta/2 \; e^{i\phi} \leftarrow \boxed{\text{Exercise 4!}}$$

$$\therefore \quad dz = \tfrac{1}{2}\sec^2\theta/2 \; e^{i\phi} \, d\theta + i\tan\theta/2 \; e^{i\phi} \, d\phi$$

$\therefore \quad dz \wedge d\bar{z}$

$$= \tfrac{1}{2}\sec^2\tfrac{\theta}{2}\, e^{i\phi}\, d\theta + i\tan\tfrac{\theta}{2}\, e^{i\phi}\, d\phi$$
$$\wedge \left(\tfrac{1}{2}\sec^2\tfrac{\theta}{2}\, e^{-i\phi}\, d\theta - i\tan\tfrac{\theta}{2}\, e^{-i\phi}\, d\phi\right)$$

$$= -i\sec^2\tfrac{\theta}{2}\tan\tfrac{\theta}{2}\, d\theta \wedge d\phi$$

$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$

$$= -i\,\frac{\tan\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\, d\theta \wedge d\phi$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= -i\,\frac{1}{\frac{1}{2}(1+\cos\theta)} \cdot \frac{\boxed{\sin\theta \quad d\theta \wedge d\phi}^{\,\omega}}{1+\cos\theta}$$

$$\therefore \quad \omega = \frac{1}{2}i\,(1+\cos\theta)^2\, dz \wedge d\bar{z}$$

$i\,dz \wedge d\bar{z}$
$= 2\,dx \wedge dy$

$\cos^2 + \sin^2 = 1$

$1 + \tan^2 = \sec^2$

$$= 2i\cos^4\left(\frac{\theta}{2}\right)\, dz \wedge d\bar{z}$$

$$= \frac{2i}{(1+|z|^2)^2}\, dz \wedge d\bar{z}$$

Since $z = \tan\frac{\theta}{2}\, e^{i\phi}$
$\therefore |z|^2 = \tan^2\frac{\theta}{2}$

$$= \frac{4}{(1+x^2+y^2)^2}\, dx \wedge dy$$

From this calculation we also see that, as expected,

$$\omega = i\partial\bar{\partial}\phi$$

for the smooth real function $\phi(z,\bar{z}) = 2\log(1+|z|^2)$.

Check: $i\partial\bar{\partial}\log(1+z\bar{z})$

$$= i\partial\left(\frac{z}{1+z\bar{z}}\,d\bar{z}\right)$$

$$= i\left(\frac{(1+z\bar{z})\cdot 1 - z\bar{z}}{(1+z\bar{z})^2}\right) dz\wedge d\bar{z}$$

$$= \frac{2i}{(1+|z|^2)^2}\, dz\wedge d\bar{z}$$

We also find one more way to compute the integral of ω over S^2, since now we can express it as an integral over $\mathbb{R}^2$:

$$\int_{S^2}\omega = i\int_{\mathbb{C}} \frac{2}{(1+|z|^2)^2}\, dz\wedge d\bar{z}$$

$$= \iint_{\mathbb{R}^2} \frac{2}{(1+x^2+y^2)^2} \, dx\,dy$$

$$= 2\pi \cdot 2 \int_{r=0}^{\infty} \frac{r\,dr}{(1+r^2)^2}$$

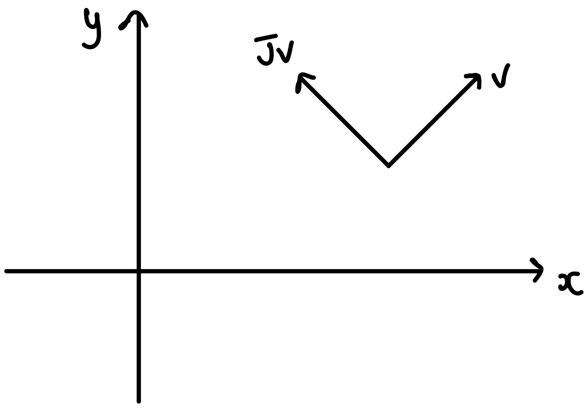
Let $u = 1+r^2$

$\therefore du = 2r\,dr$

$$= 4\pi \int_{u=1}^{\infty} \frac{du}{u^2} = -4\pi \cdot \frac{u^{-1}}{1} \Big]_1^{\infty}$$

$$= 4\pi.$$

<u>Example 3</u> $\underline{\mathbb{R}}^2 \cong \mathbb{C}$ with standard complex structure. Then the Euclidean metric g is Kähler:



$\mathbb{C}$

- $g(Jv, Jw) = g(v,w)$ ✓
- $\omega(\partial_x, \partial_y) = g(J\partial_x, \partial_y) = g(\partial_y, \partial_y) = 1$

$\therefore \; \omega = dx \wedge dy$ closed ✓

Can write this as:

$$\omega = \frac{i}{2} dz \wedge d\bar{z}$$

$$= \frac{i}{2} (dx + i dy) \wedge (dx - i dy)$$

$$= \frac{i}{2} \left[-2i \, dx \wedge dy \right]$$

$$= dx \wedge dy$$

$$\omega = i\partial\bar{\partial}\phi \quad ?$$

Yes, for $\phi = \frac{1}{2}|z|^2$.

$$= \tfrac{1}{2}\bar{z}z$$

Check: $i\partial\bar{\partial}\phi = \frac{i}{2}\partial\left(\frac{\partial}{\partial\bar{z}}(\bar{z}z)\,d\bar{z}\right)$

$$= \frac{i}{2}\partial\left(z\,d\bar{z}\right)$$

$$= \frac{i}{2}\,dz\wedge d\bar{z}$$

$$= \omega \quad \checkmark$$

Similarly, on $\mathbb{R}^{2m} \cong \mathbb{C}^m$:

$$\omega = dx_1\wedge dy_1 + \cdots + dx_m\wedge dy_m$$

$$= \frac{i}{2}\sum_{j=1}^{m} dz_j\wedge d\bar{z}_j$$

$$= i\partial\bar{\partial}\phi \qquad \phi = \frac{1}{2}\sum_{j=1}^{m}|z_j|^2.$$

Example 4 $\mathbb{CP}^n$.

We saw in example 2 that $\mathbb{CP}^1$ is a Kähler manifold. In local complex coordinates, i.e. on the chart U_0 where $w_0 \neq 0$,

$$U_0: \quad (w_0 : w_1) \longmapsto z = \frac{w_1}{w_0}$$

the Kähler potential ϕ was

$$\phi_0(z, \bar{z}) = 2\log\left(1 + |z|^2\right).$$

The factor of "2" doesn't look right from a holomorphic perspective, so let's drop it. Also, let's write it more invariantly:

$$\begin{aligned} \phi_0 : \quad U_0 &\longrightarrow \mathbb{C} \\ (w_0 : w_1) &\longmapsto \log\left(1 + \left|\frac{w_1}{w_0}\right|^2\right). \end{aligned}$$

Similarly, in the other chart U_1 where $w_1 \neq 0$, we had

$$\begin{aligned} \phi_1 : \quad U_1 &\longrightarrow \mathbb{C} \\ (w_0 : w_1) &\longmapsto \log\left(1 + \left|\frac{w_0}{w_1}\right|^2\right) \end{aligned}$$

Note that on $U_0 \cap U_1$, ϕ_0 does not agree with ϕ_1, instead:

$$\phi_1 = \phi_0 + \log\left(\left|\frac{w_0}{w_1}\right|^2\right)$$

So the functions

$$\phi_i : U_i \longrightarrow \mathbb{C}$$

don't glue together to give a globally defined function on $\mathbb{CP}^1$!

<u>However</u>, the 2-forms

$$\omega_0 = i\partial\bar{\partial}\phi_0 \quad \text{and} \quad \omega_1 = i\partial\bar{\partial}\phi_1$$

<u>do</u> agree on $U_0 \cap U_1$, since

$$\underbrace{\partial\bar{\partial}\log\left(\left|\frac{w_0}{w_1}\right|^2\right)}=0 \qquad \text{on } U_0 \cap U_1$$

$$= \partial\bar{\partial}\log\left(|z|^2\right) \qquad \text{in } U_1\text{-chart}$$

$$= \partial\left(\frac{z}{\bar{z}z}\,d\bar{z}\right)$$

$$= 0$$

Hence we do get a globally defined 2-form ω on $\mathbb{CP}^1$!

$$\omega\Big|_{U_i} = \omega_i \in \Omega^2(U_i,\mathbb{R}), \quad \omega_0\Big|_{U_0\cap U_1} = \omega_1\Big|_{U_0\cap U_1}.$$

<u>In summary</u>: the Kähler form on $\mathbb{CP}^1$ is given locally by

$$\omega = \frac{1}{(1+|z|^2)^2}\, i\,dz\wedge d\bar{z}, \quad \int_{\mathbb{CP}^1}\omega = 2\pi.$$

Similarly, on $\mathbb{CP}^n$ we have the charts

$$U_i : w_i \neq 0 \qquad (w_0 : w_1 : \cdots : w_n) \longmapsto \Big(\underbrace{\frac{w_0}{w_i}}_{z_1}, \cdots, \underbrace{\frac{w_{i-1}}{w_i}}_{z_i}, \underbrace{\frac{w_{i+1}}{w_i}}_{z_{i+1}}, \cdots, \underbrace{\frac{w_n}{w_i}}_{z_n} \Big)$$

and the Kähler potentials on each U_i are:

$$\phi_i : U_i \longrightarrow \mathbb{R}$$
$$(w_0 : w_1 : \cdots : w_n) \longmapsto \log\Big(\sum_{j=0}^{n} \Big|\frac{w_j}{w_i}\Big|^2 \Big)$$

and we define a global 2-form ω on $\mathbb{CP}^n$ via

$$\omega\big|_{U_i} := i\partial\bar{\partial}\phi_i .$$

<u>Exercise</u> 5a) Check ω is well-defined as a global 2-form

b) Compute ω in one of the charts given by the local coordinates $(z_1, \cdots, z_n)$.

(M^{2m}, ω) symplectic manifold,

$$\mathrm{vol}_\omega \in \Omega^{2m}(M)$$

$$\mathrm{vol}_\omega := \frac{\omega^m}{m!} \neq 0 \quad \text{Liouville form}$$

at each $p \in M$ $\qquad J(\partial_{x_i}$

(M^{2m}, g) oriented Riemannian manifold

$$\mathrm{vol}_g \in \Omega^{2m}(M)$$

Let $p \in M$. Let

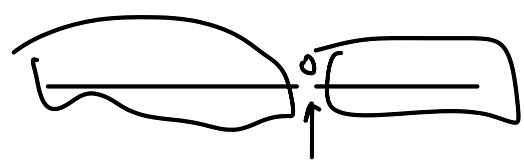
$$e_1, \cdots, e_{2m}$$

be an oriented orthonormal basis for T_pM. Has dual basis

$$e^1, \cdots, e^{2m}$$

$$\mathrm{vol}_g := e^1 \wedge \cdots \wedge e^{2m}.$$

$$\lambda \in \Lambda^{2m} T_pM \setminus \{0\}$$
$$\cong \mathbb{R} \setminus \{0\}$$



Definition An *orientation* on an n-dimensional real vector space V is a choice

$$\mathrm{or} \in \frac{\Lambda^n V}{\sim}$$

] a 2-element set, as $\Lambda^n V$ is a 1-dimensional vector space.

where $w \sim w'$ if $w = kw'$ for some $k > 0$.

We say an ordered basis $e_1, \cdots, e_n$ for V is *oriented* if

$$[e_1 \wedge \cdots \wedge e_n] = \mathrm{or} .$$

Exercise 6 Suppose that V is an n-dimensional real inner product space equipped with an orientation. Show that the formula

$$\mathrm{Vol} := e_1 \wedge \cdots \wedge e_n$$

where $e_1, \cdots, e_n$ is an oriented orthonormal basis for V, is independent of the choice of oriented orthonormal basis.

Lemma On a Kähler manifold (M, J, g, ω),

$$\mathrm{vol}_\omega = \mathrm{vol}_g$$

Proof In local coordinates $(z_1, \cdots, z_m)$ where $z_i = x_i + iy_i$, we have

$$\omega = i \sum_{j,k} h_{jk}\, dz_j \wedge d\bar{z}_k \qquad g = \sum_{j,k} h_{jk}\, dz_j \otimes d\bar{z}_k$$

where h_{ij}, $j = 1 \dots m$ is a Hermitian matrix. Now,

$$\begin{aligned}
\mathrm{Vol}_\omega &= \frac{\omega^m}{m!} \\
&= \frac{i^m}{m!} \Big(\sum_{j_1,k_1} h_{j_1 k_1}\, dz_{j_1} \wedge d\bar{z}_{k_1} \Big) \wedge \dots \wedge \Big(\sum_{j_m,k_m} h_{j_m k_m}\, dz_{j_m} \wedge d\bar{z}_{k_m} \Big) \\
&= i^m (\det h)\; dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_m \wedge d\bar{z}_m \\
&= 2^m \det h \; dx_1 \wedge dy_1 \wedge \dots \wedge dx_m \wedge dy_m
\end{aligned}$$

Exercise 7!

Use: $i\, dz \wedge d\bar{z} = 2\, dx \wedge dy$

On the other hand, an orthonormal basis for T_pM is

Exercise 8! $\rightarrow$ $e_i = \dots$, $f_i = \dots$ $\qquad i = 1 \dots m$

with dual basis

$$e^i = \dots \quad , \quad f^i = \dots$$

so that

$$\begin{aligned}
\mathrm{Vol}_g &= e^1 \wedge f^1 \wedge \dots \wedge e^m \wedge f^m \\
&= 2^m \det h \; dx_1 \wedge dy_1 \wedge \dots \wedge dx_m \wedge dy_m
\end{aligned}$$

part of exercise 8!

$$= \mathrm{vol}_\omega .$$

$\square$

3. Complex Line Bundles with Connection

3.1. Cocycle data for complex line bundles

Recall:

A <u>complex line bundle</u> over a smooth manifold M is a 1-dimensional complex vector bundle $\pi : L \longrightarrow M$.

A <u>smooth section</u> of a line bundle L is a smooth map

$$s : M \longrightarrow L$$

such that $\pi \circ s = id_M$.

Two line bundles L and L' over M are <u>isomorphic</u> if there exists a diffeomorphism ϕ making the following diagram commute

$$\begin{tikzcd} L \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & L' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

and which restricts to a linear isomorphism

$$\phi_x : L_x \longrightarrow L'_x$$

on each fiber.

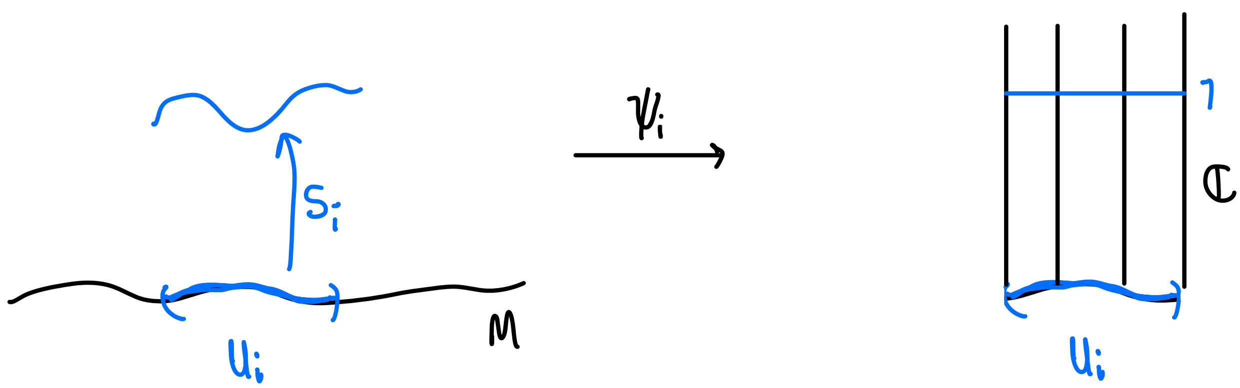
I want to phrase all of the above in terms of local data (cocycles).

Firstly, given a line bundle L over M. Let $(U_i)_{i\in I}$ be an open cover of M with local trivializations

$$\psi_i : L|_{U_i} \xrightarrow{\cong} U_i \times \mathbb{C}$$

Then for each $i \in I$, we get a <u>local section</u> $s_i \in C^\infty(U_i, L)$ by

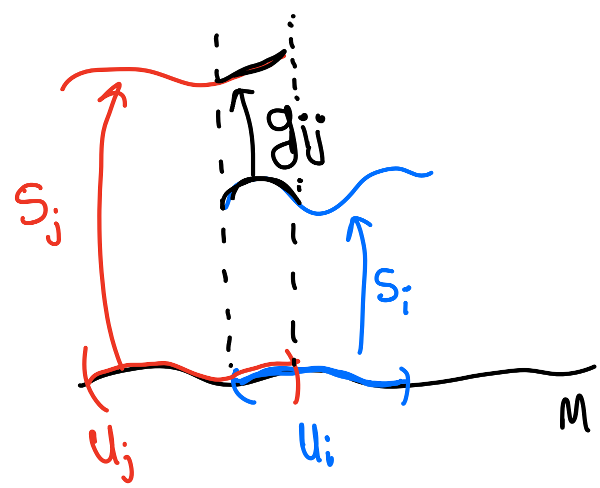
$$s_i(x) := \psi_i^{-1}(x, 1)$$



On $U_i \cap U_j$, we will have

$$s_j = g_{ij}\, s_i$$

for the <u>transition functions</u> $g_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^\times$ defined as $g_{ij} = \frac{s_j}{s_i}$.



Note that, in terms of the original local trivializations (U_i, ψ_i) of the line bundle, we can write

$$\begin{aligned} \psi_j \circ \psi_i^{-1} : U_{ij} \times \mathbb{C} &\longrightarrow U_{ij} \times \mathbb{C} \\ (x, z) &\longmapsto (x, \underbrace{g_{ij}(x)}_{\in \mathbb{C}^\times} z) \end{aligned}$$

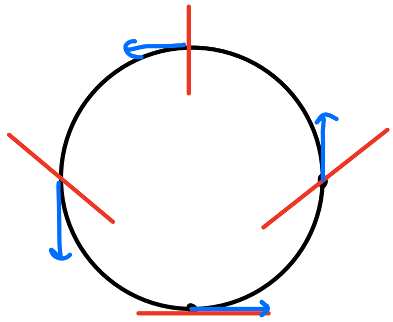
where $U_{ij} := U_i \cap U_j$ etc.

<u>Lemma</u> The transition functions (g_{ij}) satisfy the following <u>cocycle conditions</u>:

- $g_{ii} = 1$ on U_i
- $g_{ij} g_{ji} = 1$ on U_{ij}
- $g_{ij} g_{jk} g_{ki} = 1$ on U_{ijk}

<u>Exercise 1</u> Prove this.

__Example__ Consider S^1, and L = "a square root of the tangent bundle":

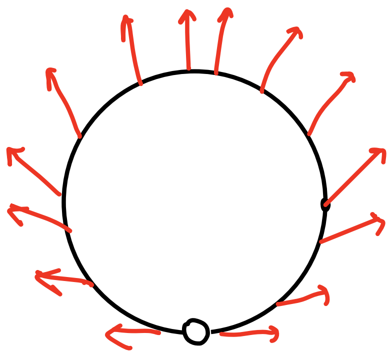


$$TS^1_{e^{i\theta}} := \mathbb{R}\left[e^{i(\theta+\pi/2)}\right]$$

$$L_{e^{i\theta}} := \mathbb{R}\left[e^{i(\theta/2+\pi/4)}\right]$$

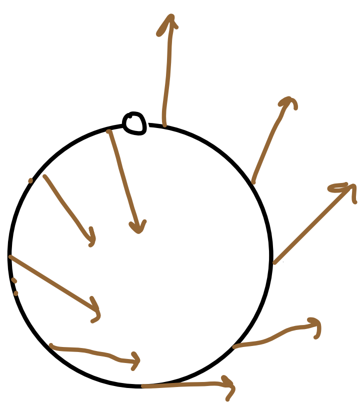
(This is a __real__ line bundle, but we can tensor it with $\mathbb{C}$ to make it a complex line bundle.)

L does not have a __global__ nonvanishing smooth section, but it does have __local__ ones:



$$U_0 = S^1 \setminus \{(0,-1)\}$$

$$s_0(e^{i\theta}) = e^{i(\theta/2+\pi/4)}$$



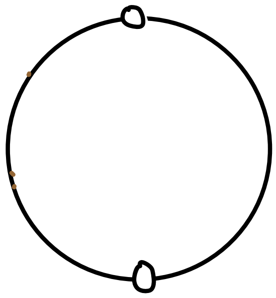
$$U_1 = S^1 \setminus \{(0,1)\}$$

$$s_0(e^{i\theta}) = e^{i(\theta/2+\pi/4)}$$

$$s_1 = \begin{cases} +1 \quad s_0 & \text{for } x>0 \\ -1 \quad s_0 & \text{for } x<0 \end{cases} \quad \text{on } U_0 \cap U_1$$

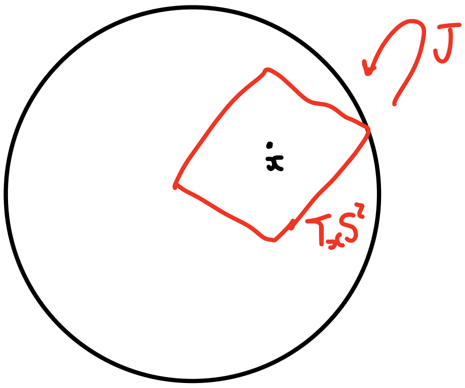
So the transition function is

$$g_{01} : U_0 \cap U_1 \longrightarrow \mathbb{C}^{\times}$$



$$g_{01} = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Example Consider the tangent bundle of S^2:



Each tangent space is naturally a 1-dimensional complex vector space using the complex structure J.

Exercise 1' Work out the transition functions of this bundle, using the "stereographic projection from north and south pole" charts.

The transition functions completely encode the line bundle up to isomorphism.

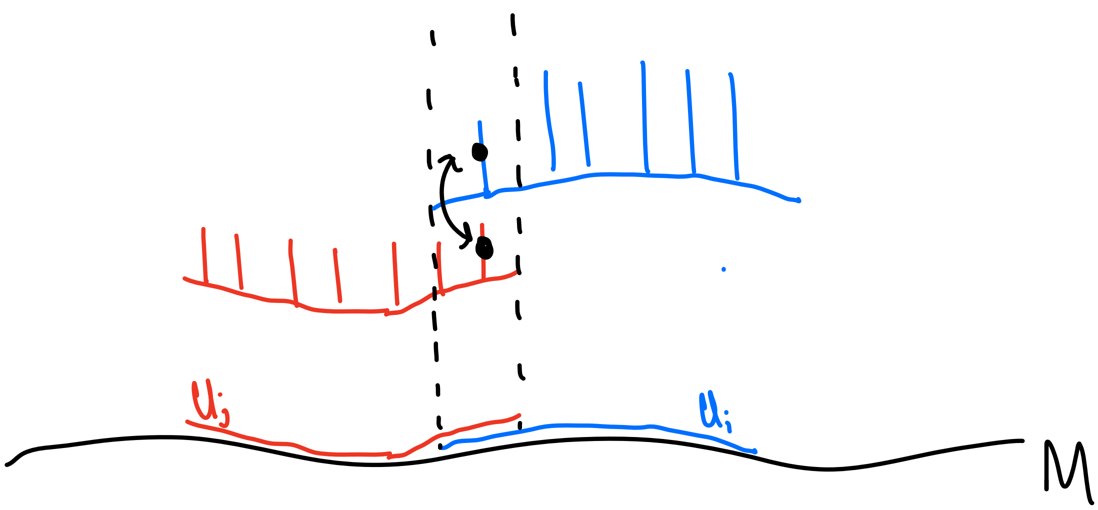
Definition Let $(M, (U_i), (g_{ij}))$ be the data of:

- a smooth manifold M
- an open covering (U_i)
- smooth functions $g_{ij} : U_{ij} \longrightarrow \mathbb{C}^\times$ satisfying the cocycle conditions

Then we define the line bundle

$$L(M, (U_i), (g_{ij})) := \coprod_{i \in I} U_i \times \mathbb{C} \Big/ \sim$$

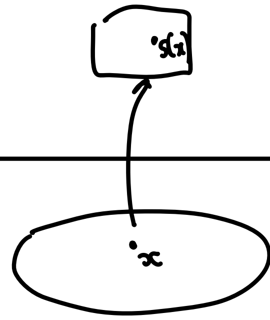
where $(x, z)_i \sim (x, g_{ij}(x) z)_j$.



Exercise: Check the validity of this construction. Where do the cocycle conditions get used?

Proposition Let $L \to M$ be a line bundle, with local trivializations (U_i, ψ_i). Then there is a canonical isomorphism

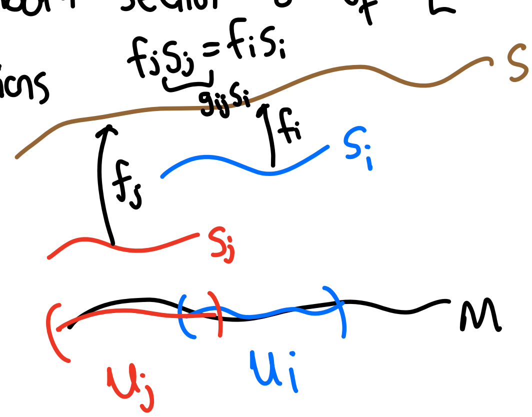
$$\begin{aligned} L &\longrightarrow L(M, U_i, g_{ij}). \\ v &\longmapsto [\psi_i(v)] \end{aligned}$$



Exercise 3 Check this!

This allows us to express **sections** of L in terms of cocycle data.

Proposition Under the above isomorphism, a smooth section s of L corresponds to a collection of smooth functions



$$f_i : U_i \longrightarrow \mathbb{C}$$

satisfying

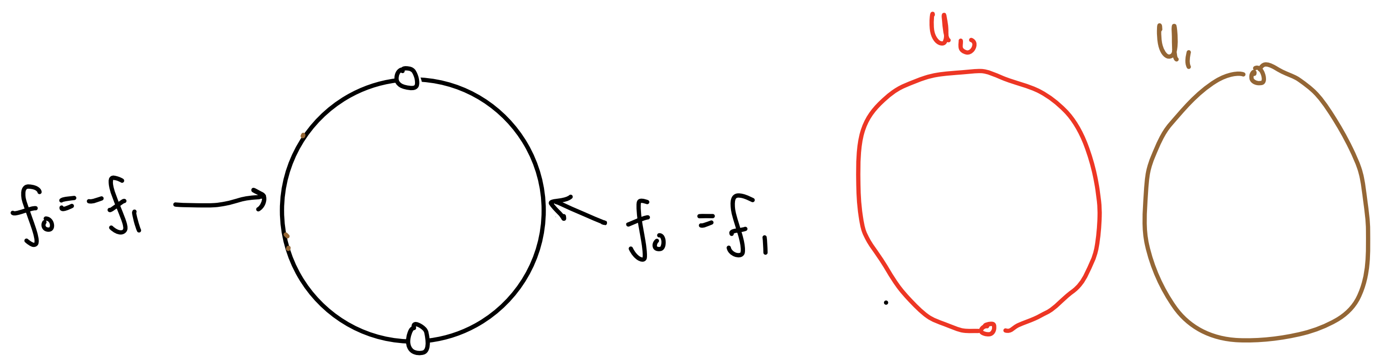
$$f_j = g_{ji} f_i \qquad \text{on } U_{ij}.$$

Exercise 4 Check this!

Example For the "square root of the tangent bundle of S^1" line bundle L earlier, a smooth section of L corresponds to a pair of smooth functions

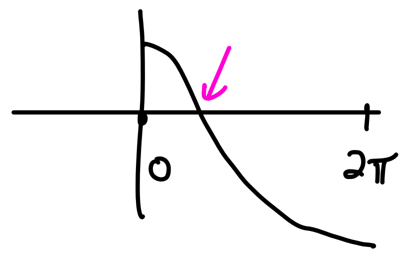
$$f_0 : U_0 \longrightarrow \mathbb{R} \quad , \quad f_1 : U_1 \longrightarrow \mathbb{R}$$

such that on $U_0 \cap U_1$,



We can say that a smooth section of L corresponds to a smooth function

$$f : [0, 2\pi] \longrightarrow \mathbb{C}$$



having antiperiodic boundary conditions, $f(2\pi) = -f(0)$.

Note that such a function must be zero somewhere, which shows that L is nontrivial (as a real bundle).

Exercise 5 Is $L \otimes \mathbb{C}$ nontrivial as a complex line bundle?

We can also say when two line bundles constructed from cocycles will be isomorphic.

<u>Lemma</u> Let $(M, (U_i), (g_{ij}))$ and $(M, (U_i), (g_{ij}'))$ be cocycle data. Then they are <u>isomorphic</u> if and only if there exist nonvanishing smooth functions

$$h_i : U_i \longrightarrow \mathbb{C}^\times$$

such that

$$g_{ij}' = \frac{h_i}{h_j} g_{ij} \quad \text{on } U_{ij}$$

<u>Exercise 6</u> Prove this!

3.2. Cohomological classification of line bundles

(See Schottenloher, Lecture notes in geometric quantization, appendix E)

Definition Let X be a topological space, and (U_i) an open cover. A <u>Čech k-cochain</u> on X with values in a discrete abelian group A is a family of locally constant functions

$$\eta = \left(\eta_{i_0 \cdots i_k} : U_{i_0} \cap \cdots \cap U_{i_k} \longrightarrow A\right)$$

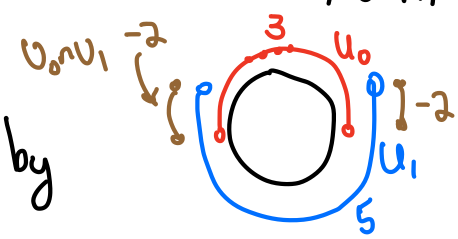
We write the collection of Čech cochains as $\check{C}^k(X, (U_i); A)$

There is a <u>coboundary map</u>

$A = \mathbb{Z}$

$$\delta : \check{C}^k(X, (U_i), A) \longrightarrow \check{C}^{k+1}(X, (U_i), A)$$

defined by



$\eta_0 : U_0 \to A$ $\quad \eta \in C^0$

$\eta_1 : U_1 \to A$ $\quad \delta\eta \in C^1$

$(\delta\eta)_{01} = (-1)^0 \eta_1 + (-1)^1 \eta_0$

$= \eta_1 - \eta_0$

$$(\delta\eta)_{i_0 \cdots i_{k+1}} := \sum_{j=0}^{k+1} (-1)^j \, \eta_{i_0 \cdots \hat{i_j} \cdots i_k}$$

which squares to zero, i.e. $\delta^2 = 0$.

<u>Exercise 1</u> Check that $\delta^2 = 0$.

We define the k^{th} Cech cohomology group of X, subordinate to the open cover (U_i), as

$$\check{H}^k(X,(U_i);A) := \frac{\operatorname{Ker}(\delta : Z^k \longrightarrow Z^{k+1})}{\operatorname{Im}(\delta : Z^{k-1} \longrightarrow Z^k)}$$

If $(V_j)_{j\in J}$ is a <u>refinement</u> of $(U_i)_{i\in I}$ [i.e. for every $j \in J$ there exists $i(j) \in I$ such that $V_j \subseteq U_{i(j)}$] then there is a natural homomorphism

$$\check{H}^k(X,(U_i);A) \longrightarrow \check{H}^k(X,(V_j);A)$$

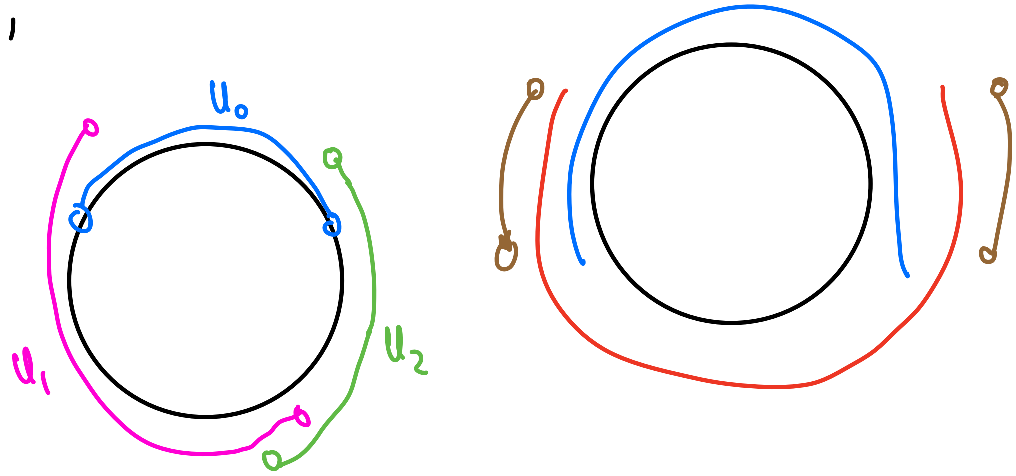
We define the k^{th} Čech cohomology group of X with coefficients in A as the direct limit, i.e.

$$\boxed{\check{H}^k(X;A) := \varinjlim_{\text{open covers } \mathcal{U}} \check{H}(X,\mathcal{U},A).}$$

Happily, if $\mathcal{U} = (U_i)$ is a <u>Leray cover</u> (i.e. all intersections $U_{i_0} \cap \cdots \cap U_{i_k}$ are contractible), then we have a natural isomorphism

$$\check{H}^k(X,\mathcal{U};A) \cong \check{H}(X;A)$$

<u>Example</u> On S^1,



is a Leray cover. A 0-cochain, with coefficients in $\mathbb{Z}$ (say), is a collection of 3 integers:

$$\eta_i \in \mathbb{Z}$$

Its coboundary is

$$\delta\eta = \begin{cases} (\delta\eta)_{01} = \eta_1 - \eta_0 & \text{on } U_{01} \\ (\delta\eta)_{02} = \eta_2 - \eta_0 & \text{on } U_{02} \\ (\delta\eta)_{12} = \eta_2 - \eta_1 & \text{on } U_{12} \end{cases}$$

So, $\delta\eta = 0 \iff \eta_0 = \eta_1 = \eta_2$.

So, $\check{H}^0(S^1; \mathbb{Z}) \cong \mathbb{Z}$.

$\boxed{\text{Exercise 2: Compute } \check{H}^1(S^1;\mathbb{Z}) \text{ in a similar way.}}$

Theorem Let M be a topological space. Then there is a natural bijection

$$\left\{ \begin{matrix} \text{Isomorphism classes of} \\ \text{complex line bundles } L \text{ over } M \end{matrix} \right\} \cong \check{H}^2(M; \mathbb{Z})$$

Proof The bijection is given at the level of cocycle data by:

$$(L, (U_i), (g_{ij})) \longmapsto (\eta_{ijk} \in \mathbb{Z} : U_i \cap U_j \cap U_k \neq \phi)$$

where η is defined as follows. On $U_i \cap U_j$ we have smooth functions

$$g_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^\times.$$

Satisfying

$$g_{ij} g_{jk} g_{ki} = 1 \qquad \text{on } U_i \cap U_j \cap U_k$$

Since $U_i \cap U_j$ is contractible, we can take the log of g_{ij} on $U_i \cap U_j$, and it will satisfy

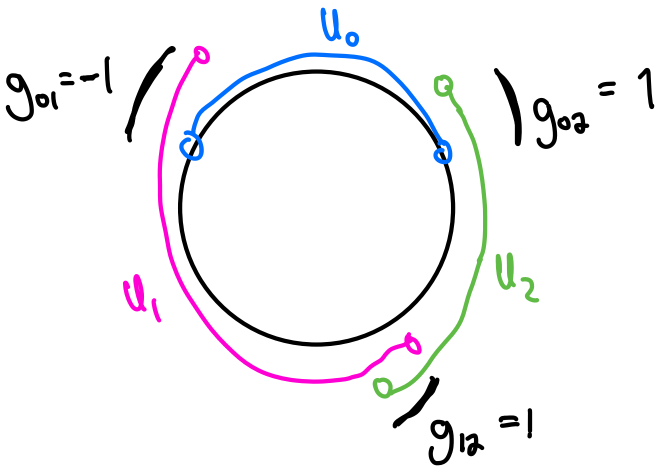
$$\log(g_{ij}) + \log(g_{jk}) + \log(g_{ki}) = n \cdot 2\pi i$$

on $U_i \cap U_j \cap U_k$. This integer $n(i,j,k)$ is our Čech cocycle! That is,

$$\eta_{ijk} = n(i,j,k) \quad \text{on} \quad U_i \cap U_j \cap U_k .$$

<u>Exercise 3</u> Fill in the rest of this proof! $\square$

<u>Example</u> For the square root line bundle on S^1,



Take $\log(g_{01}) = \pi i$

Then eg. on $U_0 \cap U_1 \cap U_1$,

$$\log(g_{01}) + \log(g_{11}) + \log(g_{10})$$

$$= \pi i \quad + \quad 0 \quad - \pi i \quad = 0$$

At any rate, $[\eta] = 0$ in $\check{H}^2(S^1; \mathbb{Z})$.

It turns out that on a smooth manifold M, Čech cohomology with coefficients in $\mathbb{R}$ is the same as De Rham cohomology!

Theorem On a smooth manifold M, there is a natural isomorphism

$$\Psi : H^k_{dR}(M;\mathbb{R}) \longrightarrow \check{H}^k(M;\mathbb{R})$$

also works over $\mathbb{C}$.

Proof We will only write down the map for $k=2$.

Choose a Leray cover (U_i) of M.

Let $W \in H^2_{dR}(M;\mathbb{R})$. Let $\omega \in \Omega^2(M,\mathbb{R})$ be a representative for W.

Since each open set U_i is contractible, we have

$$\begin{aligned} d\omega &= 0 && \text{on } U_i \\ \Rightarrow \quad \omega &= d\beta_i && \text{on } U_i \end{aligned}$$

for 1-forms $\beta_i \in \Omega^1(U_i,\mathbb{R})$. Similarly,

$$\begin{aligned} d(\beta_i - \beta_j) &= \omega - \omega && \text{on } U_i \cap U_j \\ &= 0 \\ \Rightarrow \quad \beta_i - \beta_j &= df_{ij} && \text{on } U_i \cap U_j \end{aligned}$$

for 0-forms $f_{ij} \in \Omega^0(U_i \cap U_j, \mathbb{R})$. Now,

$$d(f_{ij} + f_{jk} + f_{ki}) = 0 \quad \text{on} \quad U_i \cap U_j \cap U_k$$

and hence

$$\eta_{ijk} = f_{ij} + f_{jk} + f_{ki} \in \mathbb{R} \quad \text{is locally constant on } U_i \cap U_j \cap U_k.$$

We have $\delta \eta = 0$ on $U_i \cap U_j \cap U_k \cap U_\ell$.

Set $\Psi(W) := [\eta]$.

$\square$

Putting these two results together gives a map

$$\boxed{\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{complex line bundles } L \text{ over } M \end{array} \right\} \longrightarrow \left[\begin{array}{c} \text{integral elements in} \\ H^2_{dR}(M; \mathbb{R}) \end{array} \right] H^2_{dR}(M, \mathbb{Z})}$$

where the "integral elements" in $H^2_{dR}(M; \mathbb{R})$ are the classes of the forms such that

$$\int_\Sigma \omega \in \mathbb{Z} \qquad \text{for all oriented surfaces } \Sigma \subseteq M.$$

3.3. Holomorphic line bundles

Definition A complex line bundle $\pi: L \longrightarrow M$ over a complex manifold is a holomorphic line bundle if L is equipped with the structure of a complex manifold and π is a holomorphic map.

Recall that the cocycle data for a smooth topological line bundle L over a smooth manifold M is given by an open covering (U_i) of M and transition functions

$$g_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^\times$$

satisfying:

- $g_{ii} = 1$ on U_i
- $g_{ij} g_{ji} = 1$ on $U_i \cap U_j$
- $g_{ij} g_{jk} g_{ki} = 1$ on $U_i \cap U_j \cap U_k$.

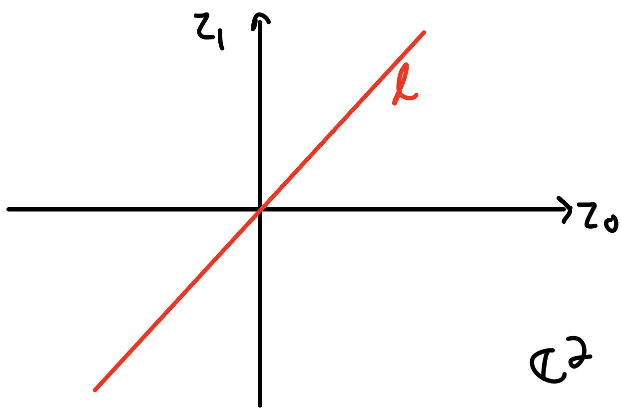
Similarly, the data of a holomorphic line bundle is the same, except that the transition functions g_{ij} must be holomorphic.

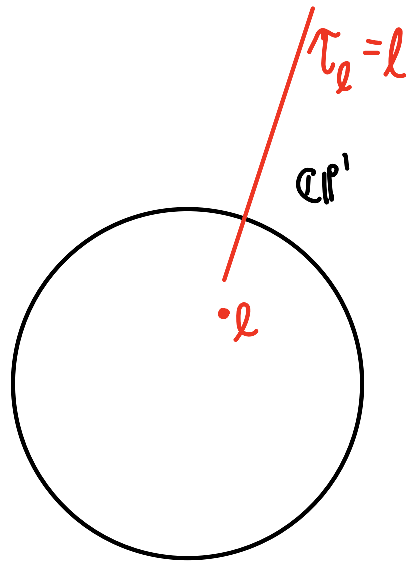
__Example__ Since

$$\mathbb{C}P^1 = \{1\text{-dimensional subspaces of } \mathbb{C}^2\},$$

there is a natural __tautological line bundle__ τ over $\mathbb{C}P^1$,

$$\tau_\ell := \ell$$





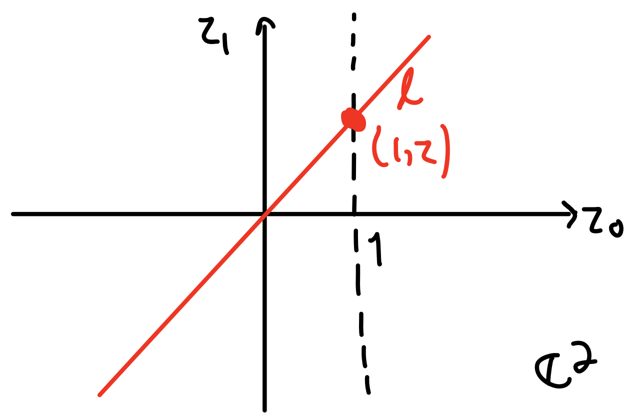
Said differently,

$$\tau = \{(\ell, v) : \ell \in \mathbb{C}P^1, \ v \in \ell\}.$$

We have local trivializations as follows. On U_0 (where $z_0 \neq 0$)

$$\psi_0 : \tau|_{U_0} \longrightarrow U_0 \times \mathbb{C}$$

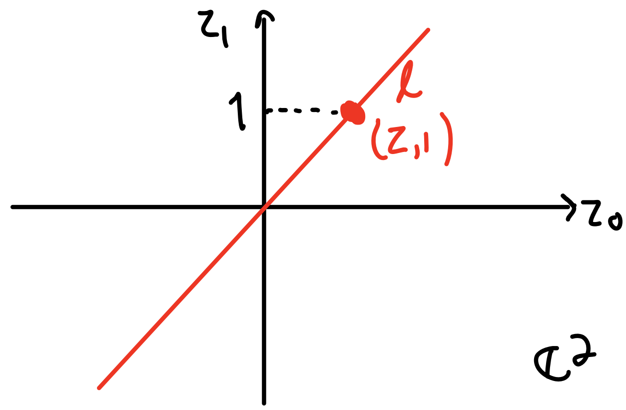
$$([1:z], \lambda(1,z)) \longmapsto ([1:z], \lambda)$$



while on U_1

$$\psi_1 : \tau|_{U_1} \longrightarrow U_1 \times \mathbb{C}$$

$$\left([w:1], \mu(z,1)\right) \longmapsto \left([w:1], \mu\right)$$



So the local sections are

$$s_0\left([1:z]\right) = (1,z) \qquad \text{on} \quad U_0$$

$$s_1\left([w:1]\right) = (w,1) \qquad \text{on} \quad U_1$$

and the transition functions are given by

$$s_1 = g_{01}\, s_0 \qquad \text{on} \quad U_0 \cap U_1$$

i.e. $$s_1([1:z]) = s_1\left(\left[\tfrac{1}{z} : 1\right]\right)$$

$$= \left(\tfrac{1}{z}, 1\right)$$

$$= \tfrac{1}{z}(1, z)$$

$$= \underbrace{\tfrac{1}{z}}_{= g_{01}([1:z])} s_0$$

This is a holomorphic function, so τ is a holomorphic line bundle.

A <u>holomorphic section</u> s of τ will take the form

$$s|_{U_0} = f_0 \, s_0 \qquad \text{on } U_0$$

$$s|_{U_1} = f_1 \, s_1 \qquad \text{on } U_1$$

And we need

$$f_0 s_0 = f_1 s_1 \qquad \text{on } U_0 \cap U_1$$

i.e.

$$\frac{f_0}{f_1} = \frac{s_1}{s_0} \qquad \text{on } U_0 \cap U_1$$

In terms of the charts

$$\begin{aligned} \phi_0 : U_0 &\xrightarrow{\cong} \mathbb{C} \\ [1:z] &\longmapsto z \end{aligned} \qquad\qquad \begin{aligned} \phi_1 : U_1 &\longrightarrow \mathbb{C} \\ [z:1] &\longmapsto z \end{aligned}$$

if we set

$$\hat{f}_i := f_i \, \phi_i^{-1}$$

we therefore need:

$$\frac{f_0}{f_1}([z_0:z_1]) = g_{01}([z_0:z_1])$$

$$= z_0/z_1 \qquad \text{on } U_0 \cap U_1$$

i.e.

$$\frac{\hat{f}_0 \circ \phi_0 \ ([z_0:z_1])}{\hat{f}_1 \circ \phi_1 \ ([z_0:z_1])} = z_0/z_1 \qquad \text{on} \qquad U_0 \cap U_1$$

i.e.

$$\frac{\hat{f}_0\left(z_1/z_0\right)}{\hat{f}_1\left(z_0/z_1\right)} = z_0/z_1 \qquad \text{on } U_0 \cap U_1$$

$$\boxed{\text{i.e.} \qquad \frac{\hat{f}_0(z)}{\hat{f}_1(1/z)} = 1/z \qquad \text{on } U_0 \cap U_1}$$

i.e. we need holomorphic functions

$$\{\hat{f}_0, \hat{f}_1 : \mathbb{C} \longrightarrow \mathbb{C} \}$$

satisfying

$$\hat{f}_1\left(\tfrac{1}{z}\right) = z\,\hat{f}_0(z). \qquad \text{on } \mathbb{C}\setminus\{0\}$$

Is this possible? Well, we can expand

$$\hat{f}_0 = a_0 + a_1 z + \cdots \qquad \hat{f}_1 = b_1 + b_1 z + \cdots$$

so we need, on $\mathbb{C}\setminus\{0\}$,

$$\begin{aligned} b_0 + b_1 z^{-1} + \cdots &= z\left(a_0 + a_1 z + \cdots\right) \\ &= a_0 z + a_1 z^2 + \cdots \end{aligned}$$

which has the unique solution $a_i = b_i = 0$ for all i.

$$\therefore \quad \mathrm{Hol}(\mathbb{CP}^1, \tau) = \{0\}.$$

On the other hand,

$$\mathrm{Hol}\left(\mathbb{CP}^1, \tau^\vee\right) \cong \left(\mathbb{C}^2\right)^\times \qquad \text{(check!)}$$

and hence is 2-dimensional.

3.3. Connections on line bundles

Definition Let $L \to M$ be a complex line bundle over a smooth manifold. A connection ∇ on L consists of the data of

$$\nabla_X s \in C^\infty(M, L)$$

for every $X \in C^\infty(M, TM)$, $s \in C^\infty(M, L)$, satisfying:

$$\text{(1)} \quad \nabla_{fX+gY}(s) = f\nabla_X s + g\nabla_Y s$$

$$\text{(2)} \quad \nabla_X(fs) = X(f)s + f\nabla_X s$$

Given local trivializations for L with accompanying non-vanishing local sections s_i on U_i, we can write

$$\nabla_X s_i = \alpha_i(X)s_i$$

for some 1-forms $\alpha_i \in \Omega^1(U_i)$. Notice that on $U_i \cap U_j$

$$s_j = g_{ij} s_i$$

$$\therefore \quad \nabla_X s_j = \nabla_X(g_{ij} s_i)$$

$$\therefore \quad \alpha_j(X)\underbrace{s_j}_{=g_{ij}s_i} = dg_{ij}(X)\, s_i + g_{ij}\,\alpha_i(X)\, s_i$$

$$\therefore \quad g_{ij}\,\alpha_j = dg_{ij} + g_{ij}\,\alpha_i \quad \text{on } U_i \cap U_j$$

$$\therefore \quad \boxed{\alpha_j = \alpha_i + \frac{dg_{ij}}{g_{ij}} \quad \text{on } U_i \cap U_j} \quad (*)$$

Conversely, given a collection of 1-forms $\alpha_i \in \Omega^1(U_i)$ satisfying $(*)$, we can construct a unique connection whose associated local 1-forms are the α_i. So:

$$\text{connection } \nabla \text{ on } (M, L) = \left\{ \begin{array}{c} \text{1-forms } \alpha_i \text{ on } U_i \text{ satisfying} \\ (*) \end{array} \right\}$$

<u>Lemma</u> The abelian group $\Omega^1(M, \mathbb{C})$ acts freely and transitively on the set $\mathrm{Conn}(L)$ of connections on L, via the formula

$$(\beta \cdot \nabla)_X(s) := \nabla_X s + \beta(X) s.$$

<u>Proof</u> Firstly, check whether $\beta \cdot \nabla$ is indeed a connection?

Satisfies (1) ? ✓

Satisfies (2) ?

Check:

$$(\beta\cdot\nabla)_X(fs) = \nabla_X(fs) + \beta(X)\, fs$$
$$= X(f)s + f\nabla_X s + \beta(X) fs$$
$$= X(f)s + f\left((\beta\cdot\nabla)_X s\right) \quad \checkmark$$

Group action ?

$$\beta'\cdot(\beta\cdot\nabla) \overset{?}{=} (\beta'+\beta)\cdot\nabla \quad ? \quad \checkmark$$

Action is free ?

Suppose $\beta\cdot\nabla = \nabla$

$\Rightarrow$ $(\beta\cdot\nabla)_X s = \nabla_X s$ for all X, s

$\Rightarrow$ $\nabla_X s + \beta(X)s = \nabla_X s$ " "

$\Rightarrow$ $\beta(X)s = 0$ for all X, s

$\Rightarrow$ $\beta = 0.$

Action is transitive ?

Given connections ∇, ∇', choose local non-vanishing sections s_i on U_i. We can write

$$\nabla_X s_i = \alpha_i(X)\, s_i$$
$$\nabla'_X s_i = \alpha_i'(X)\, s_i$$

for 1-forms α_i on U_i. Now, consider the 1-forms β_i:

$$(\nabla'_X - \nabla_X)(s_i) = \underbrace{(\alpha_i' - \alpha_i)}_{\beta_i}(X)\; s_i$$

On U_j, we know that

$$\alpha_j' = \alpha_i + d\log(g_{ij})$$
$$\alpha_j = \alpha_i + d\log(g_{ij})$$

$$\therefore \quad \beta_j = \alpha_j' - \alpha_j$$
$$= \alpha_i' - \alpha_i$$
$$= \beta_i$$

So the β_i glue together to give a globally well-defined 1-form. In other words,

$$\nabla'_X(\cdot) = \nabla_X(\cdot) + \beta(X).(\cdot)$$

i.e. $$\beta \cdot \nabla = \nabla'.$$

$\square$

Definition The *curvature* of a connection ∇ on a line bundle is

$$\mathrm{curv}(\nabla) := d\alpha \qquad \in \Omega^2(M)$$

where (α_i) are the local 1-forms for ∇ relative to a local trivialization (s_i).

What does this mean? Well, although $\alpha_i \neq \alpha_j$, since

$$\boxed{\alpha_j = \alpha_i + \frac{dg_{ij}}{g_{ij}} \quad \text{on } U_i \cap U_j}$$

we notice that

$$d\alpha_j = d\alpha_i + \underbrace{d\left(\frac{dg_{ij}}{g_{ij}}\right)}_{= \frac{g_{ij}\, d^2 g_{ij} - dg_{ij} \wedge dg_{ij}}{g_{ij}^2} = 0}$$

$$= d\alpha_i$$

so we get a <u>globally</u> well-defined 2-form $\mathrm{curv}(\nabla)$!

<u>Lemma</u> $\mathrm{curv}(\nabla)$ is a closed 2-form on M.

<u>Proof</u> Clear - because locally, $\mathrm{curv}(\nabla) = d\alpha_i$, so

$$d\,\mathrm{curv}(\nabla) = d^2\alpha_i = 0 \quad \text{on } U_i.$$

$\square$

<u>Lemma</u> The cohomology class

$$[\mathrm{curv}(\nabla)] \in H^2(M, \mathbb{C})$$

is independent of the choice of connection ∇ on L. It is called the <u>1st Chern class</u> of L in de Rham cohomology.

<u>Proof</u> If ∇' is another connection, then we know that

$$\nabla' = \nabla + \beta$$

for some 1-form β. That means, locally, in terms of the local 1-forms,

$$\nabla_X s_i = \alpha_i(X) s_i$$
$$\nabla'_X s_i = (\alpha_i + \beta)(X) s_i$$

i.e.

$$\mathrm{curv}(\nabla') = d(\alpha_i + \beta)$$
$$= d\alpha_i + d\beta \qquad \text{on } U_i$$

i.e.

$$\mathrm{curv}(\nabla') = \mathrm{curv}(\nabla) + d\beta \quad \text{on } M$$

i.e.

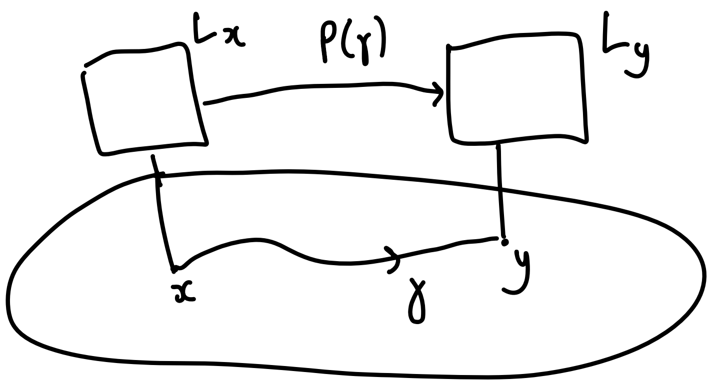
$$[\mathrm{curv}\nabla'] = [\mathrm{curv}(\nabla)]$$

$\square$

A connection ∇ on a complex line bundle L defines a <u>parallel transport</u> linear map

$$P(\gamma) : L_x \longrightarrow L_y$$

associated to any smooth curve $\gamma : [0,1] \longrightarrow M$, $\gamma(0) = x$, $\gamma(1) = y$.



How do we do this? It is defined as the solution to the ODE for a section $s(t)$ over $\gamma(t)$:

$$\nabla_{\gamma'(t)} s(t) = 0$$

In other words,

$$P(x \xrightarrow{\gamma} y) : L_x \longrightarrow L_y$$

$$v \longmapsto \text{unique soln } s(1) \text{ to ODE:}$$

$$\nabla_{\gamma'(t)} s(t) = 0.$$

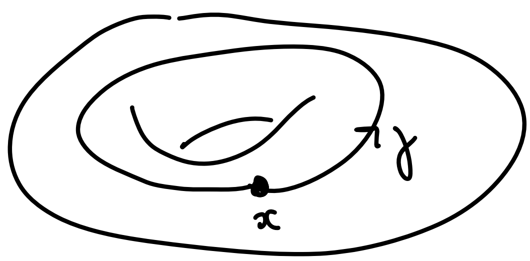
$$s(0) = v.$$

What happens when we parallel-transport around closed loops? We get a linear map

$$P(\gamma) : L_x \longrightarrow L_x$$

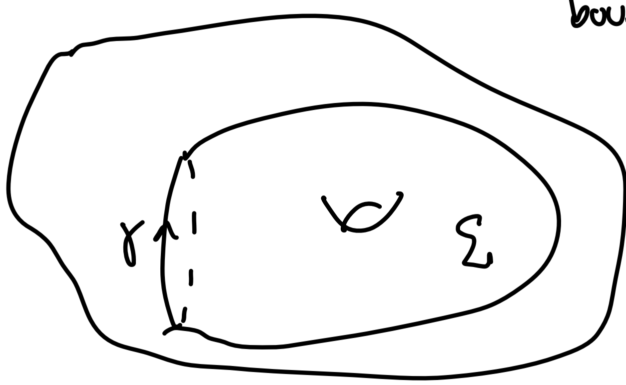
which is just multiplication by a complex number, called the holonomy of the connection around γ:

$$P(\gamma) = \mathrm{Hol}_\nabla(\gamma)\, \mathrm{id}_{L_x}.$$



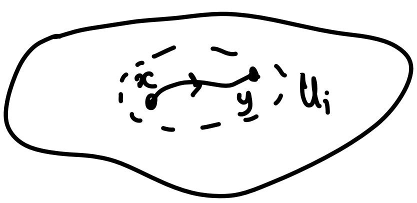
Note that $\mathrm{Hol}(\gamma)$ is independent of the basepoint x. (Why?)

Lemma $\mathrm{Hol}_\nabla(\gamma) = e^{-\int_\Sigma \mathrm{curv}(\nabla)}$ where Σ is any surface in M bounded by γ.



M

Proof Locally, $\nabla_X s_i = \alpha_i(X) s_i$, so that parallel transport in U_i is just the integral of α_i:



M

$$P(\gamma) = e^{i\int_\gamma \alpha_i}$$

Why? Well, in U_i, the ODE we must solve is

$$\nabla_{\gamma'(t)} s(t) = 0 \quad , \quad \begin{aligned} s(0) &= v \\ s(1) &= ? \end{aligned}$$

We can write

$$s(t) = e^{if(t)} s_i(t)$$

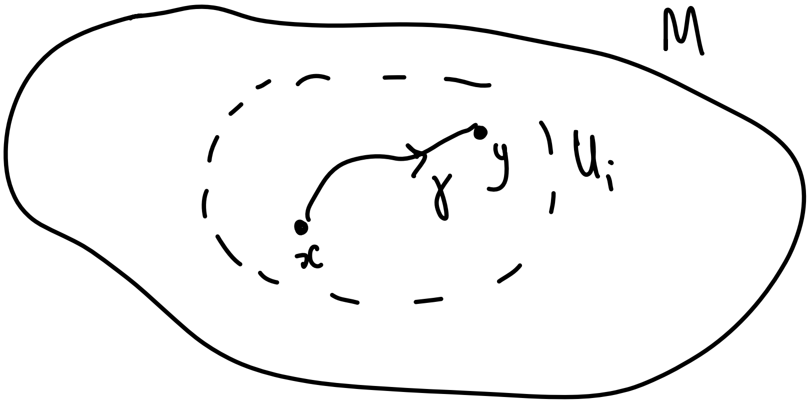
where s_i is our local section on U_i, i.e. $\nabla_X s_i = \alpha_i(X) s_i$

So our DE in terms of $f(t)$ is:

$$\begin{aligned} \nabla_{\gamma'(t)} s(t) = 0 \quad &\Leftrightarrow \quad \nabla_{\gamma'(t)} \left(e^{if} s_i \right) = 0 \\ &\Leftrightarrow \quad i e^{if} \frac{df(\gamma(t))}{dt} s_i + e^{if} \alpha_i(\gamma'(t)) s_i = 0 \\ &\Leftrightarrow \quad \frac{df(\gamma(t))}{dt} = i \alpha_i(\gamma'(t)) \\ &\Leftrightarrow \quad f(\gamma(t)) = i \int_0^t \alpha_i(\gamma'(s)) ds \\ &\Leftrightarrow \quad f(y) = f(x) + i \int_{\gamma : x \to y} \alpha_i \end{aligned}$$

In other words, in U_i, parallel transport is given by integrating α_i:

$$\boxed{\begin{aligned} P(\gamma) : L_x &\longrightarrow L_y \\ s_i(x) &\longmapsto e^{-\int_\gamma \alpha_i} s_i(y) \end{aligned}}$$



In general, we would break up γ into paths γ_i in each U_i. And then, in each U_i, we could implement Stokes lemma!!

$$\int_{\partial \Sigma_i} \alpha = \int_{\Sigma_i} d\alpha$$

which leads to the formula. (Details needed!)

$\square$

$$\boxed{\underline{\text{Corollary}}\quad \frac{1}{2\pi i}\,\mathrm{curv}(\nabla)\in\Omega^2(M,\mathbb{R})\ \text{ is an } \underline{\text{integral}}\ 2\text{-form.}}$$

Proof We need to prove that integrating the 2-form

$$2\pi i\,\mathrm{curv}(\nabla)$$

over closed surfaces Σ in M gives an integer. Well, by the previous formula for holonomy, we know:

$$e^{-\int_\Sigma \mathrm{curv}\nabla} = \underbrace{\mathrm{Hol}_\nabla(\underbrace{\partial\Sigma}_{=\phi})}_{=1}$$

$$\therefore \int_\Sigma \mathrm{curv}\nabla = n\cdot 2\pi i \quad , n\in\mathbb{Z}.$$

$\square$

We have shown (most of) the following

Theorem Given a smooth manifold M, the map

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{complex line bundles over } M \end{array} \right\} \longrightarrow H^2_{dR}(M;\mathbb{Z})$$

is given by

$$[L] \longmapsto \underbrace{\frac{1}{2\pi i}\left[\operatorname{curv}\nabla\right]}$$

↑ the 1st Chern class of L

where ∇ is any connection on L.

Moreover, this map is surjective.

Final remarks

We start with smooth manifolds.

Almost complex manifolds are nice examples of smooth manifolds.

Complex manifolds are nice examples of almost complex manifolds.

Kähler manifolds are nice examples of complex manifolds.

Integral Kähler manifolds (where the symplectic form ω has integral periods) are nice examples of Kähler manifolds!

Indeed, if our Kähler manifold (M, J, ω, g) has integral ω, then we know from the above that there exists a line bundle L with connection ∇ such that $\mathrm{curv}\nabla = -i\omega$!

So: our Kähler data (the symplectic form ω) arises from a more primitive geometric object: the line bundle L with connection!

Moreover, we have the following:

Kodaira embedding theorem A compact complex manifold M admits an embedding into projective space $\mathbb{CP}^n$ if and only if there exists a line bundle L on M with connection ∇ such that $\omega := i\,\mathrm{curv}(\nabla)$ is a Kähler form on M.

i.e.: a compact complex manifold admits an embedding into projective space

$\iff$

it admits an integral Kähler metric!

Even more is true!

Chow's theorem Every complex submanifold of projective space admits the structure of an algebraic variety.

So we have a remarkable correspondence between complex differential geometry and algebraic geometry!

Compact integral Kähler manifolds = smooth projective algebraic varieties.